EE 521 Instrumentation and Measurements
Fall 2007
Solutions for homework assignment #4

3.1

(a) The noise power spectrum of a resistor is $S_n(f) = 4kTR$. The RMS noise voltage in the 100 Hz to 20 kHz band is

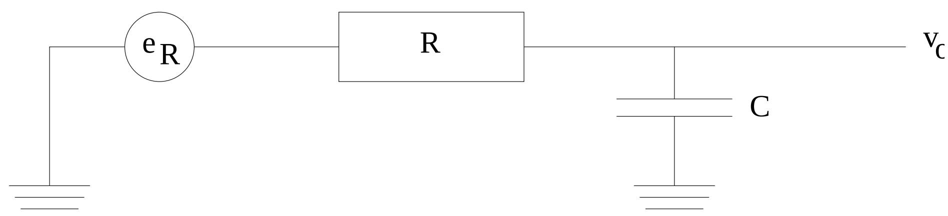
$$\langle v_n^2 \rangle = \int_{100}^{20\times10^3} 4kTRdf = 4kTR19900 = 1.656\times10^{-20}\times10^6\times19900 = 3.3\times10^{-10}\,\mathrm{V}^2$$

and

$$\sqrt{\langle v_n^2 \rangle} = 0.18\,\mu\mathrm{V}$$

(b)

The white noise spectrum is now passed through a RC low-pass filter. That looks like this



The filter transfer function is

$$H(s) = \frac{\frac{1}{sC}}{R+\frac{1}{sC}} = \frac{1}{sRC+1}$$

The white noise power spectrum passed through this filter becomes

$$\begin{aligned} S_{no}(f) =& S_n(f)\left|H(j2\pi f)\right|^2 \\ =& 4kTR\left|\frac{1}{j2\pi fRC+1}\right|^2 \\ =& 4kTR\frac{1}{1+(2\pi RCf)^2} \end{aligned}$$

(c)

The final task is to find the RMS voltage from the power spectrum S_{no}. This is

$$\langle v_{no}^2 \rangle = \int_0^\infty S_{no}(f)df$$

But first I will re-write

$$S_{no}(f) = 4kTR\frac{\frac{1}{(2\pi RC)^2}}{\frac{1}{(2\pi RC)^2} + f^2} = \frac{4kTR}{2\pi RC}\frac{\frac{1}{2\pi RC}}{\frac{1}{(2\pi RC)^2} + f^2}$$

We can now apply the definite integral given in the problem and obtain

$$\langle v_{no}^2 \rangle = \frac{4kTR}{2\pi RC}\frac{\pi}{2} = \frac{kT}{C}$$

Inserting given values we get

$$\sqrt{\langle v_{no}^2 \rangle} = \sqrt{\frac{kT}{C}} = \sqrt{\frac{1.656 \times 10^{-20}}{4 \times 23 \times 10^{-12}}} = 13.4\,\mu\text{V}$$

3.2

(a)

The output voltage is the input voltage multiplied by the amplifier gain and multiplied by the filter gain at the frequency f_0,

$$V_o(t) = V_s \sin(2\pi f_0 t)\, K_V H(j2\pi f_0)$$

$$\langle V_o^2 \rangle = V_s^2 \frac{1}{2} K_V^2 \left| H(j2\pi f_0) \right|^2$$

where

$$\left| H(j2\pi f_0) \right|^2 = \frac{1}{(2\pi\tau f_0)^2 + 1}$$

So that

$$\langle V_o^2 \rangle = \frac{V_s^2}{2}\frac{K_V^2}{(2\pi\tau f_0)^2 + 1} = \frac{V_s^2}{2}\frac{(10^3)^2}{(2\pi 0.01 \times 100)^2 + 1} = 12352 V_s^2$$

and

$$\sqrt{\langle V_o^2 \rangle} = 111 V_s = 0.0011\,\text{V}$$

(b)

A white noise spectrum is generated by the resistor. Additional white noise is added before the amplifier. Both are amplified by the amplifier and then filtered by the LPF. At the point V_i, the noise power spectrum is

$$S_i(f) = 4kTR + e_{na}^2$$

At the point V_o', the noise power spectrum is

$$S_o'(f) = \left(4kTR + e_{na}^2\right)^2 K_V^2$$

At the point V_o, the noise power spectrum is

$$
\begin{aligned}
S_o(f) =& S_o'(f)\left|H(j2\pi f)\right|^2 \\
&= \left(4kTR + e_{na}^2\right)^2 K_V^2 \left|\frac{1}{\tau j2\pi f + 1}\right|^2 \\
&= \left(4kTR + e_{na}^2\right)^2 K_V^2 \frac{1}{\left(2\pi\tau f\right)^2 + 1}
\end{aligned}
$$

Next we find the MS noise output as

$$
\begin{aligned}
\langle v_o^2\rangle =& \int_0^\infty S_o(f)df \\
&= \left(4kTR + e_{na}^2\right)^2 K_V^2 \int_0^\infty \frac{1}{\left(2\pi\tau f\right)^2 + 1}df
\end{aligned}
$$

We can look this integral up in the Table 3.1, and we get

$$
\langle v_o^2\rangle = \left(4kTR + e_{na}^2\right)^2 K_V^2 \frac{1}{4\tau}
$$

(c) Find the filter time-constant, τ_0 which will maximize the output SNR. The output SNR is

$$
\begin{aligned}
\text{SNR}_o =& \frac{\langle V_o^2\rangle}{\langle v_o^2\rangle} \\
=& \frac{V_s^2}{2}\frac{K_V^2}{\left(2\pi\tau f_0\right)^2 + 1}\frac{4\tau}{\left(4kTR + e_{na}^2\right)^2 K_V^2} \\
=& 4\frac{V_s^2}{2}\frac{1}{\left(4kTR + e_{na}^2\right)^2}\frac{\tau}{\left(2\pi\tau f_0\right)^2 + 1} \\
=& \frac{2V_s^2}{\left(4kTR + e_{na}^2\right)^2}\frac{1}{\frac{1}{\tau} + \tau\left(2\pi f_0\right)^2}
\end{aligned}
$$

To maximize SNR_o we must minimize the denominator, which is when it's derivative is zero,

$$
0 = \frac{d}{d\tau}\left(\frac{1}{\tau} + \tau\left(2\pi f_0\right)^2\right)
$$

$$
-\frac{1}{\tau_0^2} + \left(2\pi f_0\right)^2 = 0
$$

$$
\tau_0 = \frac{1}{2\pi f_0}
$$

With $f_0 = 100\,\text{Hz}$, we get $\tau_0 = 1.59 \times 10^{-3}$. Inserting this value into the expression for SNR_o, we get

$$
\begin{aligned}
\mathrm{SNR}_o =& \frac{2V_s^2}{\left(4kTR + e_{na}^2\right)^2} \frac{1}{\frac{1}{\tau} + \tau \left(2\pi f_0\right)^2} \\
=& \frac{2 \times \left(10 \times 10^{-6}\right)^2}{1.656 \times 10^{-20} \times 1.5 \times 10^3 + \left(14 \times 10^{-9}\right)^2} \times \frac{1}{\frac{1}{1.59 \times 10^{-3}} + 1.59 \times 10^{-3} \left(2\pi 100\right)^2} \\
=& 720
\end{aligned}
$$