

# EE 565: Position, Navigation and Timing

## Error Mechanization (ECEF)

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- In continuous time notation
  - Attitude:  $\dot{C}_b^e = C_b^e \Omega_{eb}^b$
  - Velocity:  $\dot{\vec{v}}_{eb}^e = C_b^e \vec{f}_{ib}^b + \vec{g}_b^e - 2\Omega_{ie}^i \vec{v}_{eb}^e$
  - Position:  $\dot{\vec{r}}_{ib}^e = \vec{v}_{eb}^e$
- In State-space notation

$$\vec{\omega}_{ib}^b = \vec{\omega}_{ie}^b + \vec{\omega}_{eb}^b$$

$$\Omega_{eb}^b = \Omega_{ib}^b - \Omega_{ie}^b$$

$$\begin{bmatrix} \dot{\vec{r}}_{eb}^e \\ \dot{\vec{v}}_{eb}^e \\ \dot{C}_b^i \end{bmatrix} = \begin{bmatrix} \vec{v}_{eb}^e \\ C_b^e \vec{f}_{ib}^b + \vec{g}_b^e - 2\Omega_{ie}^i \vec{v}_{eb}^e \\ C_b^e \Omega_{eb}^b \end{bmatrix} \quad (1)$$

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$$\begin{bmatrix} \dot{\vec{r}}_{eb}^e \\ \dot{\vec{v}}_{eb}^e \\ \dot{C}_b^e \end{bmatrix} = \begin{bmatrix} \vec{v}_{eb}^e \\ C_b^e \vec{f}_{ib}^b + \vec{g}_b^e - 2\Omega_{ie}^i \vec{v}_{eb}^e \\ C_b^e \Omega_{eb}^b \end{bmatrix} \quad (1)$$

## Question

What is the effect of sensor noise in the measurements  $\vec{\omega}_{ib}^b$  and  $\vec{f}_{ib}^b$  on the navigation solution position, velocity and attitude?

$$\dot{\hat{C}}_b^e = C_b^e \Omega_{eb}^b = C_b^e (\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt} \left[ (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \hat{C}_b^e \right] =$$

$$\dot{\hat{C}}_b^e = C_b^e \Omega_{eb}^b = C_b^e (\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt} \left[ (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \hat{C}_b^e \right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \hat{C}_b^e \Omega_{eb}^b = [\delta\dot{\vec{\psi}}_{eb}^e \times] \hat{C}_b^e + (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \dot{\hat{C}}_b^e =$$

$$\dot{\hat{C}}_b^e = C_b^e \Omega_{eb}^b = C_b^e (\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt} \left[ (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \hat{C}_b^e \right] =$$

$$\begin{aligned}
 (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \hat{C}_b^e \Omega_{eb}^b &= [\delta\dot{\vec{\psi}}_{eb}^e \times] \hat{C}_b^e + (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \dot{\hat{C}}_b^e = \\
 (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \hat{C}_b^e (\hat{\Omega}_{eb}^b + \delta\Omega_{ib}^b - \delta\Omega_{ie}^b) &=
 \end{aligned}$$

$$\dot{\hat{C}}_b^e = C_b^e \Omega_{eb}^b = C_b^e (\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt} \left[ (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \hat{C}_b^e \right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \hat{C}_b^e \Omega_{eb}^b = [\delta\dot{\vec{\psi}}_{eb}^e \times] \hat{C}_b^e + (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \dot{\hat{C}}_b^e =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \hat{C}_b^e (\hat{\Omega}_{eb}^b + \delta\Omega_{ib}^b - \delta\Omega_{ie}^b) =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \hat{C}_b^e \hat{\Omega}_{eb}^b + \hat{C}_b^e (\delta\Omega_{ib}^b - \delta\Omega_{ie}^b) =$$

$$\because [\delta\vec{\psi}_{eb}^e \times] \delta\Omega_{eb}^b \approx 0$$

$$\dot{C}_b^e = C_b^e \Omega_{eb}^b = C_b^e (\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt} \left[ (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \hat{C}_b^e \right] =$$

$$\begin{aligned}
 (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \hat{C}_b^e \Omega_{eb}^b &= [\delta\dot{\vec{\psi}}_{eb}^e \times] \hat{C}_b^e + (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \dot{\hat{C}}_b^e = \\
 (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \hat{C}_b^e (\hat{\Omega}_{eb}^b + \delta\Omega_{ib}^b - \delta\Omega_{ie}^b) &= \\
 (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \hat{C}_b^e \hat{\Omega}_{eb}^b + \hat{C}_b^e (\delta\Omega_{ib}^b - \delta\Omega_{ie}^b) &=
 \end{aligned}$$



$$\dot{C}_b^e = C_b^e \Omega_{eb}^b = C_b^e (\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt} \left[ (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \hat{C}_b^e \right] =$$

$$\begin{aligned} (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \hat{C}_b^e \Omega_{eb}^b &= [\delta\dot{\vec{\psi}}_{eb}^e \times] \hat{C}_b^e + (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \dot{\hat{C}}_b^e = \\ (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \hat{C}_b^e (\hat{\Omega}_{eb}^b + \delta\Omega_{ib}^b - \delta\Omega_{ie}^b) &= \\ (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \hat{C}_b^e \hat{\Omega}_{eb}^b + \hat{C}_b^e (\delta\Omega_{ib}^b - \delta\Omega_{ie}^b) &= \end{aligned}$$

$$[\delta\dot{\vec{\psi}}_{eb}^e \times] = \hat{C}_b^e (\delta\Omega_{ib}^b - \delta\Omega_{ie}^b) \hat{C}_e^b = [\hat{C}_b^e (\delta\vec{\omega}_{ib}^b - \delta\vec{\omega}_{ie}^b) \times] \quad (2)$$

$$\dot{C}_b^e = C_b^e \Omega_{eb}^b = C_b^e (\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt} \left[ (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \hat{C}_b^e \right] =$$

$$\begin{aligned} (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \hat{C}_b^e \Omega_{eb}^b &= [\delta\dot{\vec{\psi}}_{eb}^e \times] \hat{C}_b^e + (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \dot{\hat{C}}_b^e = \\ (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \hat{C}_b^e (\hat{\Omega}_{eb}^b + \delta\Omega_{ib}^b - \delta\Omega_{ie}^b) &= \\ (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]) \hat{C}_b^e \hat{\Omega}_{eb}^b + \hat{C}_b^e (\delta\Omega_{ib}^b - \delta\Omega_{ie}^b) &= \end{aligned}$$

$$[\delta\dot{\vec{\psi}}_{eb}^e \times] = \hat{C}_b^e (\delta\Omega_{ib}^b - \delta\Omega_{ie}^b) \hat{C}_e^b = [\hat{C}_b^e (\delta\vec{\omega}_{ib}^b - \delta\vec{\omega}_{ie}^b) \times] \quad (2)$$

$$\delta\dot{\vec{\psi}}_{eb}^e = \hat{C}_b^e (\delta\omega_{ib}^b - \delta\omega_{ie}^b) \quad (3)$$

$$\begin{aligned}
 \dot{\delta\vec{\psi}}_{eb}^e &= \hat{C}_b^e(\delta\vec{\omega}_{ib}^b - \delta\vec{\omega}_{ie}^b) \\
 &= \hat{C}_b^e\delta\vec{\omega}_{ib}^b - \hat{C}_b^e(\vec{\omega}_{ie}^b - \hat{\vec{\omega}}_{ie}^b) \\
 &= \hat{C}_b^e\delta\vec{\omega}_{ib}^b - (\hat{C}_b^e C_e^b - \mathcal{I})\vec{\omega}_{ie}^e \\
 &= \hat{C}_b^e\delta\vec{\omega}_{ib}^b + \delta\vec{\psi}_{eb}^e \times \vec{\omega}_{ie}^e
 \end{aligned}$$

$$\begin{aligned}
 \delta \dot{\vec{\psi}}_{eb}^e &= \hat{C}_b^e (\delta \vec{\omega}_{ib}^b - \delta \vec{\omega}_{ie}^b) \\
 &= \hat{C}_b^e \delta \vec{\omega}_{ib}^b - \hat{C}_b^e (\vec{\omega}_{ie}^b - \hat{\vec{\omega}}_{ie}^b) \\
 &= \hat{C}_b^e \delta \vec{\omega}_{ib}^b - (\hat{C}_b^e C_e^b - \mathcal{I}) \vec{\omega}_{ie}^e \\
 &= \hat{C}_b^e \delta \vec{\omega}_{ib}^b + \delta \vec{\psi}_{eb}^e \times \vec{\omega}_{ie}^e
 \end{aligned}$$

$$\delta \dot{\vec{\psi}}_{eb}^e = \hat{C}_b^e \delta \vec{\omega}_{ib}^b - \vec{\Omega}_{ie}^e \delta \vec{\psi}_{eb}^e \quad (4)$$

$$\dot{\vec{v}}_{eb}^e = C_b^e \vec{f}_{ib}^b + \vec{g}_b^e - 2\Omega_{ie}^e \vec{v}_{eb}^e \quad (5)$$

$$\begin{aligned} \hat{\dot{\vec{v}}}_{eb}^e &= \hat{C}_b^e \hat{\vec{f}}_{ib}^b + \hat{\vec{g}}_b^e - 2\Omega_{ie}^e \hat{\vec{v}}_{eb}^e \\ &= (\mathcal{I} - [\delta\vec{\psi}_{eb}^e \times]) C_b^e (\vec{f}_{ib}^b - \delta\vec{f}_{ib}^b) + \hat{\vec{g}}_b^e - 2\Omega_{ie}^e \hat{\vec{v}}_{eb}^e \end{aligned} \quad (6)$$

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$$\begin{aligned} \delta\dot{\vec{v}}_{eb}^e &= \dot{\vec{v}}_{eb}^e - \hat{\dot{\vec{v}}}_{eb}^e = [\delta\vec{\psi}_{eb}^e \times] C_b^e \vec{f}_{ib}^b + \hat{C}_b^e \delta\vec{f}_{ib}^b + \delta\vec{g}_b^e - 2\Omega_{ie}^e \delta\vec{v}_{eb}^e \\ &= [\delta\vec{\psi}_{eb}^e \times] \hat{C}_b^e \hat{\vec{f}}_{ib}^b + \hat{C}_b^e \delta\vec{f}_{ib}^b + \delta\vec{g}_b^e - 2\Omega_{ie}^e \delta\vec{v}_{eb}^e \end{aligned}$$

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$$\begin{aligned} \delta\dot{\vec{v}}_{eb}^e &= \dot{\vec{v}}_{eb}^e - \hat{\dot{\vec{v}}}_{eb}^e = [\delta\vec{\psi}_{eb}^e \times] C_b^e \vec{f}_{ib}^b + \hat{C}_b^e \delta\vec{f}_{ib}^b + \delta\vec{g}_b^e - 2\Omega_{ie}^e \delta\vec{v}_{eb}^e \\ &= [\delta\vec{\psi}_{eb}^e \times] \hat{C}_b^e \hat{\vec{f}}_{ib}^b + \hat{C}_b^e \delta\vec{f}_{ib}^b + \delta\vec{g}_b^e - 2\Omega_{ie}^e \delta\vec{v}_{eb}^e \end{aligned}$$

$$\delta\dot{\vec{v}}_{eb}^e = -[\hat{C}_b^e \hat{\vec{f}}_{ib}^b \times] \delta\vec{\psi}_{eb}^e + \hat{C}_b^e \delta\vec{f}_{ib}^b + \delta\vec{g}_b^e - 2\Omega_{ie}^e \delta\vec{v}_{eb}^e \quad (7)$$

Using Taylor series expansion, the gravity error as a function of position estimates and errors is derived to be

$$\delta \vec{g}_b^e \approx \frac{2g_0(\hat{L}_b)}{r_{eS}^e(\hat{L}_b)} \frac{\hat{\vec{r}}_{eb}^e}{|\hat{\vec{r}}_{eb}^e|^2} (\hat{\vec{r}}_{eb}^e)^T \delta \vec{r}_{eb}^e \quad (8)$$



$$\dot{\vec{r}}_{eb}^e = \vec{v}_{eb}^e \quad (9)$$

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$$\delta \dot{\vec{r}}_{eb}^e = \delta \vec{v}_{eb}^e \quad (10)$$

$$\begin{pmatrix} \delta\dot{\psi}_{eb}^e \\ \delta\dot{\vec{v}}_{eb}^e \\ \delta\dot{\vec{r}}_{eb}^e \end{pmatrix} = \begin{bmatrix} -\Omega_{ie}^e & 0_{3 \times 3} & 0_{3 \times 3} \\ -[\hat{C}_b^e \hat{\vec{r}}_{ib}^b \times] & -2\Omega_{ie}^e & \frac{2g_0(\hat{L}_b)}{r_{eS}^e(\hat{L}_b)} \frac{\hat{\vec{r}}_{eb}^e}{|\hat{\vec{r}}_{eb}^e|^2} (\hat{\vec{r}}_{eb}^e)^T \\ 0_{3 \times 3} & \mathcal{I}_{3 \times 3} & 0_{3 \times 3} \end{bmatrix} \begin{pmatrix} \delta\psi_{eb}^e \\ \delta\vec{v}_{eb}^e \\ \delta\vec{r}_{eb}^e \end{pmatrix} + \begin{bmatrix} 0 & \hat{C}_b^e \\ \hat{C}_b^e & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \delta\vec{f}_{ib}^b \\ \delta\vec{\omega}_{ib}^b \end{pmatrix} \quad (11)$$

- Truth value

 $\vec{x}$ 

- Measured value

 $\tilde{\vec{x}}$ 

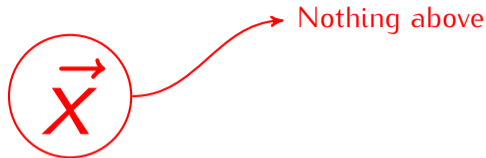
- Estimated or computed value

 $\hat{\vec{x}}$ 

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value
- Measured value
- Estimated or computed value
- Error


 $\vec{x}$ 
 $\hat{\vec{x}}$ 

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

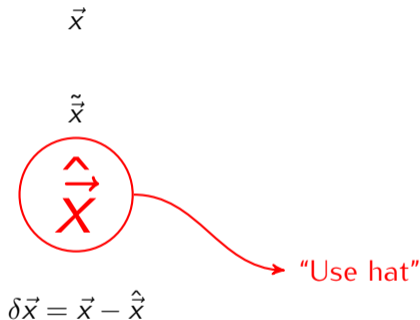
- Truth value
- Measured value
- Estimated or computed value
- Error



$\hat{\vec{x}}$

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value
- Measured value
- Estimated or computed value
- Error



- Truth value

$\vec{x}$

- Measured value

$\tilde{\vec{x}}$

- Estimated or computed value

$\hat{\vec{x}}$

- Error

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Given a non-linear system  $\dot{\vec{x}} = f(\vec{x}, t)$

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Let's assume we have an estimate of  $\vec{x}$ , i.e.,  $\hat{\vec{x}}$  such that  $\vec{x} = \hat{\vec{x}} + \delta\vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \quad (12)$$

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Using Taylor series expansion

$$\begin{aligned} f(\hat{\vec{x}} + \delta\vec{x}, t) = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} &= f(\hat{\vec{x}}, t) + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T \\ &\approx \dot{\hat{\vec{x}}} + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \end{aligned}$$

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Using Taylor series expansion

$$\begin{aligned} f(\hat{\vec{x}} + \delta\vec{x}, t) &= \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = \underbrace{f(\hat{\vec{x}}, t)}_{=} + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T \\ &\approx \underbrace{\dot{\hat{\vec{x}}}}_{=} + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \end{aligned}$$

Given a non-linear system  $\dot{\vec{x}} = f(\vec{x}, t)$

Let's assume we have an estimate of  $\vec{x}$ , i.e.,  $\hat{\vec{x}}$  such that  $\vec{x} = \hat{\vec{x}} + \delta\vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \quad (12)$$

Using Taylor series expansion

$$\begin{aligned} f(\hat{\vec{x}} + \delta\vec{x}, t) = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} &= f(\hat{\vec{x}}, t) + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T \\ &\approx \dot{\hat{\vec{x}}} + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \end{aligned}$$

$$\Rightarrow \delta\dot{\vec{x}} \approx \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \quad (13)$$

Initially the accelerometer and gyroscope measurements,  $\tilde{\vec{f}}_{ib}^b$  and  $\tilde{\vec{\omega}}_{ib}^b$ , respectively, will be modeled as

$$\tilde{\vec{f}}_{ib}^b = \vec{f}_{ib}^b + \Delta\vec{f}_{ib}^b \quad (14)$$

$$\tilde{\vec{\omega}}_{ib}^b = \vec{\omega}_{ib}^b + \Delta\vec{\omega}_{ib}^b \quad (15)$$

where  $\vec{f}_{ib}^b$  and  $\vec{\omega}_{ib}^b$  are the specific force and angular rates, respectively; and  $\Delta\vec{f}_{ib}^b$  and  $\Delta\vec{\omega}_{ib}^b$  represents the errors. In later lectures we will discuss more detailed description of these errors.

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$$\tilde{\vec{f}}_{ib}^b = \vec{f}_{ib}^b + \Delta\vec{f}_{ib}^b \quad \text{these terms may} \quad (14)$$

$$\tilde{\vec{\omega}}_{ib}^b = \vec{\omega}_{ib}^b + \Delta\vec{\omega}_{ib}^b \quad \text{be expanded further} \quad (15)$$

where  $\vec{f}_{ib}^b$  and  $\vec{\omega}_{ib}^b$  are the specific force and angular rates, respectively; and  $\Delta\vec{f}_{ib}^b$  and  $\Delta\vec{\omega}_{ib}^b$  represents the errors. In later lectures we will discuss more detailed description of these errors.

## Accelerometers

$$\tilde{\vec{f}}_{ib}^b = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^b + \vec{n}l_a + \vec{w}_a \quad (16)$$

## Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^b = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^b + G_g\vec{f}_{ib}^b + \vec{w}_g \quad (17)$$



## Accelerometers

$$\tilde{\vec{f}}_{ib}^b = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^b + \vec{n}l_a + \vec{w}_a \quad (16)$$

Biases

## Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^b = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^b + G_g\vec{f}_{ib}^b + \vec{w}_g \quad (17)$$

## Accelerometers

$$\tilde{\vec{f}}_{ib}^b = \vec{b}_a + (\mathcal{I} + \mathbf{M}_a) \vec{f}_{ib}^b + \vec{n}l_a + \vec{w}_a \quad (16)$$

Misalignment and SF Errors



## Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^b = \vec{b}_g + (\mathcal{I} + \mathbf{M}_g) \vec{\omega}_{ib}^b + G_g \vec{f}_{ib}^b + \vec{w}_g \quad (17)$$

## Accelerometers

$$\tilde{\vec{f}}_{ib}^b = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^b + \vec{n}l_a + \vec{w}_a \quad (16)$$

Non-linearity



## Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^b = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^b + G_g\vec{f}_{ib}^b + \vec{w}_g \quad (17)$$

## Accelerometers

$$\tilde{\vec{f}}_{ib}^b = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^b + \vec{n}l_a + \vec{w}_a \quad (16)$$

G-Sensitivity

## Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^b = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^b + \underbrace{G_g \vec{f}_{ib}^b}_{\text{G-Sensitivity}} + \vec{w}_g \quad (17)$$

## Accelerometers

$$\tilde{\vec{f}}_{ib}^b = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^b + n\vec{l}_a + \vec{w}_a \quad (16)$$

Noise

## Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^b = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^b + G_g\vec{f}_{ib}^b + \vec{w}_g \quad (17)$$

- Position error

$$\delta \vec{r}_{\beta b}^{\gamma} = \vec{r}_{\beta b}^{\gamma} - \hat{\vec{r}}_{\beta b}^{\gamma} \quad (18)$$

- Velocity error

$$\delta \vec{v}_{\beta b}^{\gamma} = \vec{v}_{\beta b}^{\gamma} - \hat{\vec{v}}_{\beta b}^{\gamma} \quad (19)$$

- Specific force errors

$$\delta \vec{f}_{ib}^b = \vec{f}_{ib}^b - \hat{\vec{f}}_{ib}^b \quad (20)$$

$$\Delta_e \vec{f}_{ib}^b = \Delta \vec{f}_{ib}^b - \Delta \hat{\vec{f}}_{ib}^b = -\delta \vec{f}_{ib}^b \quad (21)$$

- Angular rate errors

$$\delta \vec{\omega}_{ib}^b = \vec{\omega}_{ib}^b - \hat{\vec{\omega}}_{ib}^b \quad (22)$$

$$\Delta_e \vec{\omega}_{ib}^b = \Delta \vec{\omega}_{ib}^b - \Delta \hat{\vec{\omega}}_{ib}^b = -\delta \vec{\omega}_{ib}^b \quad (23)$$

Define

$$\delta C_b^\gamma = C_b^\gamma \hat{C}_\gamma^b = e^{[\delta \vec{\psi}_{\gamma b}^\gamma \times]} \approx \mathcal{I} + [\delta \vec{\psi}_{\gamma b}^\gamma \times] \quad (24)$$

This is the error in attitude resulting from errors in estimating the angular rates.

The attitude error is a multiplicative small angle transformation from the actual frame to the computed frame

$$\hat{C}_b^\gamma = (\mathcal{I} - [\delta\vec{\psi}_{\gamma b}^\gamma \times]) C_b^\gamma \quad (25)$$

Similarly,

$$C_b^\gamma = (\mathcal{I} + [\delta\vec{\psi}_{\gamma b}^\gamma \times]) \hat{C}_b^\gamma \quad (26)$$



Similarly we can attempt to estimate the specific force and angular rate by applying correction based on our estimate of the error.

$$\hat{\vec{f}}_{ib}^b = \tilde{\vec{f}}_{ib}^b - \Delta \hat{\vec{f}}_{ib}^b \quad (27)$$

$$\hat{\vec{\omega}}_{ib}^b = \tilde{\vec{\omega}}_{ib}^b - \Delta \hat{\vec{\omega}}_{ib}^b \quad (28)$$

where  $\hat{\vec{f}}_{ib}^b$  and  $\hat{\vec{\omega}}_{ib}^b$  are the accelerometer and gyroscope estimated calibration values, respectively.