

# EE 565: Position, Navigation and Timing

## Error Mechanization (NAV)

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$$\dot{C}_b^n = C_b^n(\Omega_{ib}^b - \Omega_{in}^b) = \frac{d}{dt} \left[ (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n \right] =$$

$$\dot{\hat{C}}_b^n = C_b^n(\Omega_{ib}^b - \Omega_{in}^b) = \frac{d}{dt} \left[ (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n \right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n \Omega_{nb}^b = [\delta\dot{\vec{\psi}}_{nb}^n \times] \hat{C}_b^n + (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \dot{\hat{C}}_b^n =$$

$$\dot{\hat{C}}_b^n = C_b^n(\Omega_{ib}^b - \Omega_{in}^b) = \frac{d}{dt} \left[ (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n \right] =$$

$$\begin{aligned}
 (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n \Omega_{nb}^b &= [\delta\dot{\vec{\psi}}_{nb}^n \times] \hat{C}_b^n + (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \dot{\hat{C}}_b^n = \\
 &\approx (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n (\hat{\Omega}_{nb}^b + \delta\Omega_{ib}^b - \delta\Omega_{in}^b)
 \end{aligned}$$

$$\dot{\hat{C}}_b^n = C_b^n(\Omega_{ib}^b - \Omega_{in}^b) = \frac{d}{dt} \left[ (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n \right] =$$

$$\begin{aligned} (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n \Omega_{nb}^b &= [\delta\dot{\vec{\psi}}_{nb}^n \times] \hat{C}_b^n + (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \dot{\hat{C}}_b^n = \\ &\approx (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n (\hat{\Omega}_{nb}^b + \delta\Omega_{ib}^b - \delta\Omega_{in}^b) \\ &\approx (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n \hat{\Omega}_{nb}^b + \hat{C}_b^n (\delta\Omega_{ib}^b - \delta\Omega_{in}^b) \end{aligned} \quad \because [\delta\vec{\psi}_{nb}^n \times] \delta\Omega_{nb}^b \approx 0$$

$$\dot{\hat{C}}_b^n = C_b^n(\Omega_{ib}^b - \Omega_{in}^b) = \frac{d}{dt} \left[ (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n \right] =$$

$$\begin{aligned}
 (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n \Omega_{nb}^b &= [\delta\dot{\vec{\psi}}_{nb}^n \times] \hat{C}_b^n + (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \dot{\hat{C}}_b^n = \\
 &\approx (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n (\hat{\Omega}_{nb}^b + \delta\Omega_{ib}^b - \delta\Omega_{in}^b) \\
 &\approx (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n \hat{\Omega}_{nb}^b + \hat{C}_b^n (\delta\Omega_{ib}^b - \delta\Omega_{in}^b)
 \end{aligned}$$

$$\dot{\hat{C}}_b^n = C_b^n(\Omega_{ib}^b - \Omega_{in}^b) = \frac{d}{dt} \left[ (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n \right] =$$

$$\begin{aligned} (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n \Omega_{nb}^b &= [\delta\dot{\vec{\psi}}_{nb}^n \times] \hat{C}_b^n + (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \dot{\hat{C}}_b^n = \\ &\approx (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n (\hat{\Omega}_{nb}^b + \delta\Omega_{ib}^b - \delta\Omega_{in}^b) \\ &\approx (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n \hat{\Omega}_{nb}^b + \hat{C}_b^n (\delta\Omega_{ib}^b - \delta\Omega_{in}^b) \end{aligned}$$

$$[\delta\dot{\vec{\psi}}_{nb}^n \times] = \hat{C}_b^n (\delta\Omega_{nb}^b - \delta\Omega_{in}^b) \hat{C}_b^n = [\hat{C}_b^n (\delta\vec{\omega}_{ib}^b - \delta\vec{\omega}_{in}^b) \times] \quad (1)$$

$$\dot{\hat{C}}_b^n = C_b^n(\Omega_{ib}^b - \Omega_{in}^b) = \frac{d}{dt} \left[ (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n \right] =$$

$$\begin{aligned} (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n \Omega_{nb}^b &= [\delta\dot{\vec{\psi}}_{nb}^n \times] \hat{C}_b^n + (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \dot{\hat{C}}_b^n = \\ &\approx (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n (\hat{\Omega}_{nb}^b + \delta\Omega_{ib}^b - \delta\Omega_{in}^b) \\ &\approx (\mathcal{I} + [\delta\vec{\psi}_{nb}^n \times]) \hat{C}_b^n \hat{\Omega}_{nb}^b + \hat{C}_b^n (\delta\Omega_{ib}^b - \delta\Omega_{in}^b) \end{aligned}$$

$$[\delta\dot{\vec{\psi}}_{nb}^n \times] = \hat{C}_b^n (\delta\Omega_{nb}^b - \delta\Omega_{in}^b) \hat{C}_n^b = [\hat{C}_b^n (\delta\vec{\omega}_{ib}^b - \delta\vec{\omega}_{in}^b) \times] \quad (1)$$

$$\delta\dot{\vec{\psi}}_{nb}^n = \hat{C}_b^n (\delta\omega_{ib}^b - \delta\omega_{in}^b) \quad (2)$$



Recall that

$$\vec{\omega}_{in}^b = C_n^b \vec{\omega}_{in}^n$$

Expressing the above equation in terms of estimates we get

$$\begin{aligned} \hat{\vec{\omega}}_{in}^b + \delta\vec{\omega}_{in}^b &= \hat{C}_n^b (\mathcal{I} - [\delta\vec{\psi}_{nb}^n \times]) (\hat{\vec{\omega}}_{in}^n + \delta\vec{\omega}_{in}^n) \\ \delta\vec{\omega}_{in}^b &\approx \hat{C}_n^b (\delta\vec{\omega}_{in}^n - [\delta\vec{\psi}_{nb}^n \times] \hat{\vec{\omega}}_{in}^n) \end{aligned}$$

Substituting this result in Equation 2

$$\dot{\delta\vec{\psi}}_{nb}^n = -\hat{\vec{\Omega}}_{in}^n \delta\vec{\psi}_{nb}^n + \hat{C}_b^n \delta\vec{\omega}_{ib}^b - \delta\vec{\omega}_{in}^n \quad (3)$$

Using Taylor series and retaining the first order terms

$$\delta\vec{\omega}_{in}^n = \frac{\partial\hat{\omega}_{in}^n}{\partial\hat{\vec{r}}_{eb}^n} \delta\vec{r}_{eb}^n + \frac{\partial\hat{\omega}_{in}^n}{\partial\hat{\vec{v}}_{eb}^n} \delta\vec{v}_{eb}^n$$

where  $\vec{r}_{eb}^n = [L_b, \lambda_b, h_b]^T$  and

$$\hat{\omega}_{in}^n = \hat{\omega}_{ie}^n + \hat{\omega}_{en}^n = \begin{pmatrix} \omega_{ie} \cos \hat{L}_b + \frac{\hat{v}_{eb,E}^n}{R_E(\hat{L}_b) + \hat{h}_b} \\ -\frac{\hat{v}_{eb,N}^n}{R_N(\hat{L}_b) + \hat{h}_b} \\ -\omega_{ie} \sin \hat{L}_b - \frac{\hat{v}_{eb,E}^n \tan \hat{L}_b}{R_E(\hat{L}_b) + \hat{h}_b} \end{pmatrix}$$

# NAV Attitude Error (cont.)

Define terms for simplicity

$$-\hat{\Omega}_{in}^n = -[(\hat{\omega}_{ie}^n + \hat{\omega}_{en}^n) \times] = F_{\psi v} \quad (4)$$

$$-\frac{\partial \hat{\omega}_{in}^n}{\partial \hat{v}_{eb}^n} = \begin{pmatrix} 0 & -\frac{1}{R_E(\hat{L}_b) + \hat{h}_b} & 0 \\ \frac{1}{R_N(\hat{L}_b) + \hat{h}_b} & 0 & 0 \\ 0 & \frac{\tan \hat{L}_b}{R_E(\hat{L}_b) + \hat{h}_b} & 0 \end{pmatrix} = F_{\psi v} \quad (5)$$

$$-\frac{\partial \hat{\omega}_{in}^n}{\partial \hat{r}_{eb}^n} = \begin{pmatrix} \omega_{ie} \sin \hat{L}_b & 0 & \frac{\hat{v}_{eb,E}^n}{(R_E(\hat{L}_b) + \hat{h}_b)^2} \\ 0 & 0 & -\frac{\hat{v}_{eb,N}^n}{(R_N(\hat{L}_b) + \hat{h}_b)^2} \\ \omega_{ie} \cos \hat{L}_b + \frac{\hat{v}_{eb,E}^n}{(R_E(\hat{L}_b) + \hat{h}_b) \cos^2 \hat{L}_b} & 0 & -\frac{\hat{v}_{eb,E}^n \tan \hat{L}_b}{(R_E(\hat{L}_b) + \hat{h}_b)^2} \end{pmatrix} = F_{\psi r} \quad (6)$$

$$\dot{\vec{\psi}}_{nb}^n = F_{\psi\psi} \delta\vec{\psi}_{nb}^n + F_{\psi v} \delta\vec{v}_{eb}^n + F_{\psi r} \delta\vec{r}_{eb}^n + \hat{C}_b^n \delta\vec{\omega}_{ib}^b \quad (7)$$

where  $F_{\psi\psi}$ ,  $F_{\psi v}$  and  $F_{\psi r}$  are defined by Equations 4, 5 and 6, respectively.

$$\dot{\vec{v}}_{eb}^n = C_b^n \vec{f}_{ib}^b + \vec{g}_b^n - (\Omega_{en}^n + 2\Omega_{ie}^n) \vec{v}_{eb}^e \quad (8)$$

$$\begin{aligned} \dot{\hat{\vec{v}}}_{eb}^n &= \hat{C}_b^n \hat{\vec{f}}_{ib}^b + \hat{\vec{g}}_b^n - (\hat{\Omega}_{en}^n + 2\hat{\Omega}_{ie}^n) \hat{\vec{v}}_{eb}^n \\ &= (\mathcal{I} - [\delta\vec{\psi}_{nb}^n \times]) C_b^n (\vec{f}_{ib}^b - \delta\vec{f}_{ib}^b) + \hat{\vec{g}}_b^n - (\hat{\Omega}_{en}^n + 2\hat{\Omega}_{ie}^n) \hat{\vec{v}}_{eb}^n \end{aligned} \quad (9)$$

Computing the  $\delta\dot{\vec{v}}_{eb}^n$  we obtain

$$\begin{aligned} \delta\dot{\vec{v}}_{eb}^n &= \dot{\vec{v}}_{eb}^n - \dot{\hat{\vec{v}}}_{eb}^n \\ &= [\delta\vec{\psi}_{nb}^n \times] C_b^n \vec{f}_{ib}^b + \hat{C}_b^n \delta\vec{f}_{ib}^b + \delta\vec{g}_b^n + \\ &\quad - (\hat{\Omega}_{en}^n + 2\hat{\Omega}_{ie}^n) \delta\vec{v}_{eb}^n - (\delta\Omega_{en}^n + 2\delta\Omega_{ie}^n) \hat{\vec{v}}_{eb}^n \end{aligned}$$

# Velocity Cont.

$$-(\hat{\Omega}_{en}^n + 2\hat{\Omega}_{ie}^n)$$

The term  $-(\hat{\Omega}_{en}^n + 2\hat{\Omega}_{ie}^n)$  is derived as

$$-[(\hat{\omega}_{en}^n + 2\hat{\omega}_{ie}^n) \times] = \begin{pmatrix} 0 & -2\omega_{ie} \sin \hat{L}_b - \frac{\hat{v}_{eb,E}^n \tan \hat{L}_b}{R_E(\hat{L}_b) + \hat{h}_b} & \frac{\hat{v}_{eb,N}^n}{R_N(\hat{L}_b) + \hat{h}_b} \\ 2\omega_{ie} \sin \hat{L}_b + \frac{\hat{v}_{eb,E}^n \tan \hat{L}_b}{R_E(\hat{L}_b) + \hat{h}_b} & 0 & 2\omega_{ie} \cos \hat{L}_b + \frac{\hat{v}_{eb,E}^n}{R_E(\hat{L}_b) + \hat{h}_b} \\ -\frac{\hat{v}_{eb,N}^n}{R_N(\hat{L}_b) + \hat{h}_b} & -2\omega_{ie} \cos \hat{L}_b - \frac{\hat{v}_{eb,E}^n}{R_E(\hat{L}_b) + \hat{h}_b} & 0 \end{pmatrix} = F_{\Omega\Omega} \quad (10)$$

$$-(\delta\hat{\Omega}_{en}^n + 2\delta\hat{\Omega}_{ie}^n)\hat{\mathbf{v}}_{eb}^n$$

The term  $-(\delta\hat{\Omega}_{en}^n + 2\delta\hat{\Omega}_{ie}^n)\hat{\mathbf{v}}_{eb}^n$  is derived as

$$-(\delta\hat{\Omega}_{en}^n + 2\delta\hat{\Omega}_{ie}^n)\hat{\mathbf{v}}_{eb}^n = \frac{\partial F_{\Omega\Omega}}{\partial \hat{\mathbf{r}}_{eb}^n} \delta \hat{\mathbf{r}}_{eb}^n \hat{\mathbf{v}}_{eb}^n + \frac{\partial F_{\Omega\Omega}}{\partial \hat{\mathbf{v}}_{eb}^n} \delta \hat{\mathbf{v}}_{eb}^n \hat{\mathbf{v}}_{eb}^n \quad (11)$$

$$\delta \vec{g}_b^n \approx \frac{2g_0(\hat{L}_b)}{r_{eS}^e(\hat{L}_b)} \delta h_b \quad (12)$$



$$\delta \dot{\vec{v}}_{eb}^n = F_{v\psi} \delta \vec{\psi}_{nb}^n + F_{vv} \delta \vec{v}_{eb}^n + F_{vr} \delta \vec{r}_{eb}^n + \hat{C}_b^n \delta \vec{f}_{ib}^b \quad (13)$$

where  $F_{v\psi}$ ,  $F_{vv}$  and  $F_{vr}$  are defined by Equations 14, 15 and 16, respectively.

$$F_{v\psi} = - \left[ (\hat{C}_b^n \hat{f}_{ib}^b) \times \right] \quad (14)$$

$$F_{vv} = \begin{pmatrix} \frac{\hat{v}_{eb,D}^n}{(R_N(\hat{L}_b) + \hat{h}_b)} & -\frac{2\hat{v}_{eb,E}^n \tan \hat{L}_b}{(R_E(\hat{L}_b) + \hat{h}_b)} - 2\omega_{ie} \sin \hat{L}_b & \frac{\hat{v}_{eb,N}^n}{R_N(\hat{L}_b) + \hat{h}_b} \\ \frac{\hat{v}_{eb,E}^n \tan \hat{L}_b}{(R_E(\hat{L}_b) + \hat{h}_b)} + 2\omega_{ie} \sin \hat{L}_b & \frac{\hat{v}_{eb,N}^n \tan \hat{L}_b + \hat{v}_{eb,D}^n}{R_N(\hat{L}_b) + \hat{h}_b} & \frac{\hat{v}_{eb,E}^n}{(R_E(\hat{L}_b) + \hat{h}_b)} + 2\omega_{ie} \cos \hat{L}_b \\ -\frac{2\hat{v}_{eb,N}^n}{R_N(\hat{L}_b) + \hat{h}_b} & -\frac{2\hat{v}_{eb,E}^n}{(R_E(\hat{L}_b) + \hat{h}_b)} - 2\omega_{ie} \cos \hat{L}_b & 0 \end{pmatrix} \quad (15)$$

$$F_{vr} = \begin{pmatrix} -\frac{(\hat{v}_{eb,E}^n)^2 \sec^2 \hat{L}_b}{R_E(\hat{L}_b) + \hat{h}_b} - 2\hat{v}_{eb,E}^n \omega_{ie} \cos \hat{L}_b & 0 & \frac{(\hat{v}_{eb,E}^n)^2 \tan \hat{L}_b}{(R_E(\hat{L}_b) + \hat{h}_b)^2} - \frac{\hat{v}_{eb,N}^n \hat{v}_{eb,D}^n}{(R_N(\hat{L}_b) + \hat{h}_b)^2} \\ \frac{\hat{v}_{eb,N}^n \hat{v}_{eb,E}^n \sec^2 \hat{L}_b}{R_E(\hat{L}_b) + \hat{h}_b} + 2\hat{v}_{eb,N}^n \omega_{ie} \cos \hat{L}_b - 2\hat{v}_{eb,D}^n \omega_{ie} \sin \hat{L}_b & 0 & -\frac{\hat{v}_{eb,N}^n \hat{v}_{eb,E}^n \tan \hat{L}_b + \hat{v}_{eb,E}^n \hat{v}_{eb,D}^n}{(R_E(\hat{L}_b) + \hat{h}_b)^2} \\ 2\hat{v}_{eb,E}^n \omega_{ie} \sin \hat{L}_b & 0 & \left( \frac{(\hat{v}_{eb,E}^n)^2}{(R_E(\hat{L}_b) + \hat{h}_b)^2} + \frac{(\hat{v}_{eb,N}^n)^2}{(R_N(\hat{L}_b) + \hat{h}_b)^2} - \frac{2g_0(\hat{L}_b)}{r_{es}^2(\hat{L}_b)} \right) \end{pmatrix} \quad (16)$$

$$\dot{\hat{\vec{r}}}_{eb}^n = \begin{pmatrix} \frac{\hat{v}_{eb,N}^n}{R_N(\hat{L}_b) + \hat{h}_b} \\ \frac{\hat{v}_{eb,E}^n}{\cos \hat{L}_b (R_E(\hat{L}_b) + \hat{h}_b)} \\ -\hat{v}_{eb,D}^n \end{pmatrix} \quad (17)$$

Computing  $\delta \dot{\vec{r}}_{eb}^n$  using Taylor series expansion and retaining only the first order terms

$$\delta \dot{\vec{r}}_{eb}^n = F_{r\psi} \delta \vec{\psi}_{nb}^n + F_{rv} \delta \vec{v}_{eb}^n + F_{rr} \delta \vec{r}_{eb}^n \quad (18)$$

where  $F_{r\psi}$ ,  $F_{rv}$  and  $F_{rr}$  are defined by Equations 19, 20 and 21, respectively.

$$F_{r\psi} = 0_{3 \times 3} \quad (19)$$

$$F_{rv} = \begin{pmatrix} \frac{1}{R_N(\hat{L}_b) + \hat{h}_b} & 0 & 0 \\ 0 & \frac{1}{(R_E(\hat{L}_b) + \hat{h}_b) \cos \hat{L}_b} & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (20)$$

$$F_{rr} = \begin{pmatrix} 0 & 0 & -\frac{\hat{v}_{eb,N}^n}{(R_N(\hat{L}_b) + \hat{h}_b)^2} \\ \frac{\hat{v}_{eb,E}^n \tan \hat{L}_b}{(R_N(\hat{L}_b) + \hat{h}_b) \cos \hat{L}_b} & 0 & -\frac{\hat{v}_{eb,E}^n}{(R_N(\hat{L}_b) + \hat{h}_b)^2 \cos \hat{L}_b} \\ 0 & 0 & 0 \end{pmatrix} \quad (21)$$

$$\begin{pmatrix} \delta \dot{\psi}_{nb}^n \\ \delta \dot{\vec{v}}_{nb}^n \\ \delta \dot{\vec{r}}_{nb}^n \end{pmatrix} = \begin{pmatrix} F_{\psi\psi} & F_{\psi v} & F_{\psi r} \\ F_{v\psi} & F_{vv} & F_{vr} \\ 0_{3 \times 3} & F_{rv} & F_{rr} \end{pmatrix} \begin{pmatrix} \delta \psi_{nb}^n \\ \delta \vec{v}_{nb}^n \\ \delta \vec{r}_{nb}^n \end{pmatrix} + \begin{pmatrix} 0 & \hat{C}_b^n \\ \hat{C}_b^n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta \vec{f}_{ib}^b \\ \delta \vec{\omega}_{ib}^b \end{pmatrix} \quad (22)$$

- Truth value

 $\vec{x}$ 

- Measured value

 $\tilde{\vec{x}}$ 

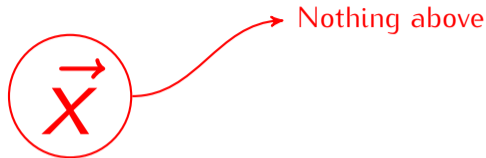
- Estimated or computed value

 $\hat{\vec{x}}$ 

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value
- Measured value
- Estimated or computed value
- Error


 $\vec{x}$ 
 $\hat{\vec{x}}$ 

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value
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- Estimated or computed value
- Error

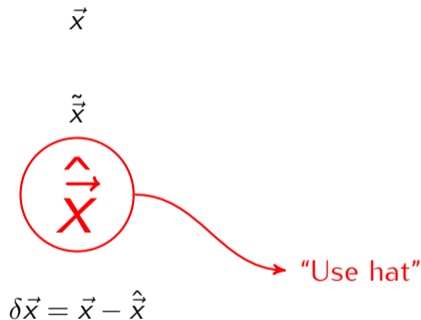


$\hat{\vec{x}}$

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$



- Truth value
- Measured value
- Estimated or computed value
- Error



- Truth value

$\vec{x}$

- Measured value

$\tilde{\vec{x}}$

- Estimated or computed value

$\hat{\vec{x}}$

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

Given a non-linear system  $\dot{\vec{x}} = f(\vec{x}, t)$

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Let's assume we have an estimate of  $\vec{x}$ , i.e.,  $\hat{\vec{x}}$  such that  $\vec{x} = \hat{\vec{x}} + \delta\vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \quad (23)$$

Given a non-linear system  $\dot{\vec{x}} = f(\vec{x}, t)$

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$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \quad (23)$$

Using Taylor series expansion

$$\begin{aligned} f(\hat{\vec{x}} + \delta\vec{x}, t) = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} &= f(\hat{\vec{x}}, t) + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T \\ &\approx \dot{\hat{\vec{x}}} + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \end{aligned}$$

Given a non-linear system  $\dot{\vec{x}} = f(\vec{x}, t)$

Let's assume we have an estimate of  $\vec{x}$ , i.e.,  $\hat{\vec{x}}$  such that  $\vec{x} = \hat{\vec{x}} + \delta\vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \quad (23)$$

Using Taylor series expansion

$$\begin{aligned} f(\hat{\vec{x}} + \delta\vec{x}, t) &= \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = \underbrace{f(\hat{\vec{x}}, t)}_{=} + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T \\ &\approx \underbrace{\dot{\hat{\vec{x}}}}_{=} + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \end{aligned}$$

Given a non-linear system  $\dot{\vec{x}} = f(\vec{x}, t)$

Let's assume we have an estimate of  $\vec{x}$ , i.e.,  $\hat{\vec{x}}$  such that  $\vec{x} = \hat{\vec{x}} + \delta\vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \quad (23)$$

Using Taylor series expansion

$$\begin{aligned} f(\hat{\vec{x}} + \delta\vec{x}, t) = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} &= f(\hat{\vec{x}}, t) + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T \\ &\approx \dot{\hat{\vec{x}}} + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \end{aligned}$$

$$\Rightarrow \delta\dot{\vec{x}} \approx \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \quad (24)$$

Initially the accelerometer and gyroscope measurements,  $\tilde{\vec{f}}_{ib}^b$  and  $\tilde{\vec{\omega}}_{ib}^b$ , respectively, will be modeled as

$$\tilde{\vec{f}}_{ib}^b = \vec{f}_{ib}^b + \Delta\vec{f}_{ib}^b \quad (25)$$

$$\tilde{\vec{\omega}}_{ib}^b = \vec{\omega}_{ib}^b + \Delta\vec{\omega}_{ib}^b \quad (26)$$

where  $\vec{f}_{ib}^b$  and  $\vec{\omega}_{ib}^b$  are the specific force and angular rates, respectively; and  $\Delta\vec{f}_{ib}^b$  and  $\Delta\vec{\omega}_{ib}^b$  represents the errors. In later lectures we will discuss more detailed description of these errors.



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$$\tilde{\vec{f}}_{ib}^b = \vec{f}_{ib}^b + \Delta\vec{f}_{ib}^b \quad \text{these terms may} \quad (25)$$

$$\tilde{\vec{\omega}}_{ib}^b = \vec{\omega}_{ib}^b + \Delta\vec{\omega}_{ib}^b \quad \text{be expanded further} \quad (26)$$

where  $\vec{f}_{ib}^b$  and  $\vec{\omega}_{ib}^b$  are the specific force and angular rates, respectively; and  $\Delta\vec{f}_{ib}^b$  and  $\Delta\vec{\omega}_{ib}^b$  represents the errors. In later lectures we will discuss more detailed description of these errors.

## Accelerometers

$$\tilde{\vec{f}}_{ib}^b = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^b + \vec{n}l_a + \vec{w}_a \quad (27)$$

## Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^b = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^b + G_g\vec{f}_{ib}^b + \vec{w}_g \quad (28)$$

## Accelerometers

$$\tilde{\vec{f}}_{ib}^b = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^b + \vec{n}l_a + \vec{w}_a \quad (27)$$

Biases



## Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^b = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^b + G_g\vec{f}_{ib}^b + \vec{w}_g \quad (28)$$

## Accelerometers

$$\tilde{\vec{f}}_{ib}^b = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^b + \vec{n}l_a + \vec{w}_a \quad (27)$$

Misalignment and SF Errors



## Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^b = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^b + G_g\vec{f}_{ib}^b + \vec{w}_g \quad (28)$$

## Accelerometers

$$\tilde{\vec{f}}_{ib}^b = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^b + \vec{n}l_a + \vec{w}_a \quad (27)$$

Non-linearity



## Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^b = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^b + G_g\vec{f}_{ib}^b + \vec{w}_g \quad (28)$$

## Accelerometers

$$\tilde{\vec{f}}_{ib}^b = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^b + \vec{n}l_a + \vec{w}_a \quad (27)$$

G-Sensitivity

## Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^b = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^b + \underbrace{G_g \vec{f}_{ib}^b}_{\text{G-Sensitivity}} + \vec{w}_g \quad (28)$$

## Accelerometers

$$\tilde{\vec{f}}_{ib}^b = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^b + n\vec{l}_a + \vec{w}_a \quad (27)$$

Noise

## Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^b = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^b + G_g\vec{f}_{ib}^b + \vec{w}_g \quad (28)$$

- Position error

$$\delta \vec{r}_{\beta b}^{\gamma} = \vec{r}_{\beta b}^{\gamma} - \hat{\vec{r}}_{\beta b}^{\gamma} \quad (29)$$

- Velocity error

$$\delta \vec{v}_{\beta b}^{\gamma} = \vec{v}_{\beta b}^{\gamma} - \hat{\vec{v}}_{\beta b}^{\gamma} \quad (30)$$

- Specific force errors

$$\delta \vec{f}_{ib}^b = \vec{f}_{ib}^b - \hat{\vec{f}}_{ib}^b \quad (31)$$

$$\Delta_e \vec{f}_{ib}^b = \Delta \vec{f}_{ib}^b - \Delta \hat{\vec{f}}_{ib}^b = -\delta \vec{f}_{ib}^b \quad (32)$$

- Angular rate errors

$$\delta \vec{\omega}_{ib}^b = \vec{\omega}_{ib}^b - \hat{\vec{\omega}}_{ib}^b \quad (33)$$

$$\Delta_e \vec{\omega}_{ib}^b = \Delta \vec{\omega}_{ib}^b - \Delta \hat{\vec{\omega}}_{ib}^b = -\delta \vec{\omega}_{ib}^b \quad (34)$$



Define

$$\delta C_b^\gamma = C_b^\gamma \hat{C}_\gamma^b = e^{[\delta \vec{\psi}_{\gamma b}^\gamma \times]} \approx \mathcal{I} + [\delta \vec{\psi}_{\gamma b}^\gamma \times] \quad (35)$$

This is the error in attitude resulting from errors in estimating the angular rates.

The attitude error is a multiplicative small angle transformation from the actual frame to the computed frame

$$\hat{C}_b^\gamma = (\mathcal{I} - [\delta\vec{\psi}_{\gamma b}^\gamma \times]) C_b^\gamma \quad (36)$$

Similarly,

$$C_b^\gamma = (\mathcal{I} + [\delta\vec{\psi}_{\gamma b}^\gamma \times]) \hat{C}_b^\gamma \quad (37)$$

Similarly we can attempt to estimate the specific force and angular rate by applying correction based on our estimate of the error.

$$\hat{\vec{f}}_{ib}^b = \tilde{\vec{f}}_{ib}^b - \Delta \hat{\vec{f}}_{ib}^b \quad (38)$$

$$\hat{\vec{\omega}}_{ib}^b = \tilde{\vec{\omega}}_{ib}^b - \Delta \hat{\vec{\omega}}_{ib}^b \quad (39)$$

where  $\hat{\vec{f}}_{ib}^b$  and  $\hat{\vec{\omega}}_{ib}^b$  are the accelerometer and gyroscope estimated calibration values, respectively.