Lecture

Navigation Mathematics: Kinematics (Earth Surface & Gravity Models)

EE 565: Position, Navigation and Timing

Lecture Notes Update on Spring 2023

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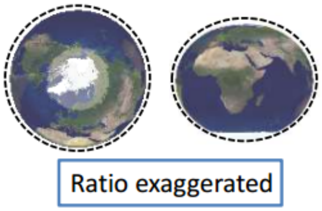
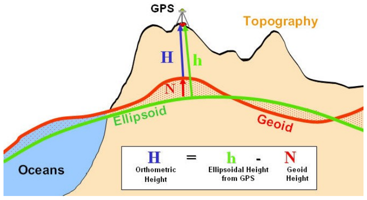
Earth Modeling

- The earth can be modeled as an oblate spheroid
  - A circular cross section when viewed from the polar axis (top view)
  - An elliptical cross-section when viewed perpendicular to the polar axis (side view)



- This ellipsoid (i.e., oblate spheroid) is an approximation of the "geoid"
- The geoid is a gravitational equipotential surface which "best" fits (in the least square sense) the mean sea level



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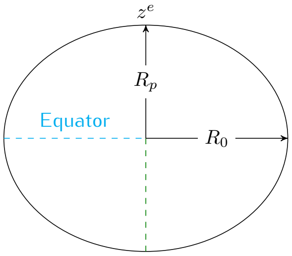
Earth Modeling

- WGS 84 provides as model of the earth's geoid
  - More recently replace by EGM 2008
- The equatorial radius radius $R_0 = 6,378,137.0\text{m}$
- The polar radius radius $R_p = 6,356,752.3142\text{m}$
- Eccentricity of the ellipsoid

$$e = \sqrt{1 - \frac{R_p^2}{R_0^2}} \approx 0.0818$$

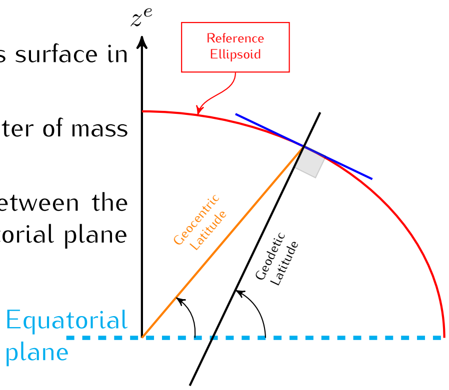
- Flattening of the ellipsoid

$$f = \frac{R_0 - R_p}{R_0} \approx \frac{1}{298}$$



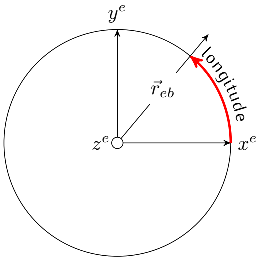
Earth Modeling

- We can define a position “near” the earth’s surface in terms of latitude, longitude, and height
    - Geocentric latitude intersects the center of mass of the earth
    - Geodetic latitude (L) is the angle between the normal to the ellipsoid and the equatorial plane



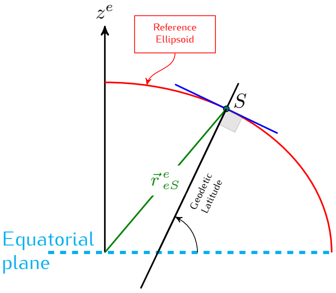
Earth Modeling

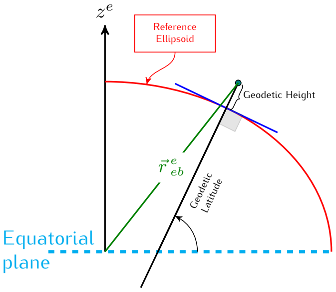
- The longitude (λ) is the angle from the x-axis of the ECEF frame to the projection of $\vec{r}_{eb}$ onto the equatorial plane



Earth Modeling

- The geocentric radius is the distance from center of the Earth to the point S
- The geodetic (or ellipsoidal) height (h) is the distance along the normal from the ellipsoid to the body





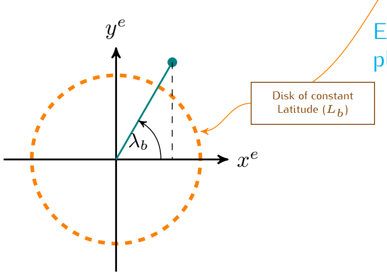
Earth Modeling

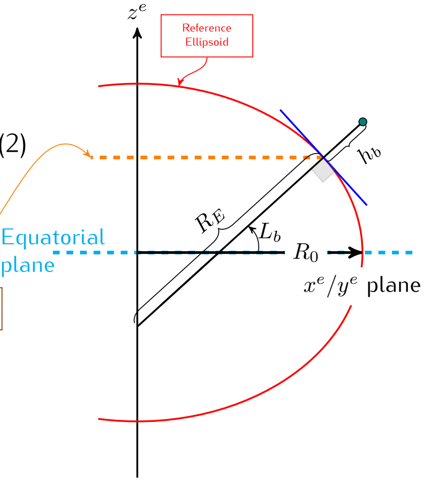
- Transverse radius of curvature

$$R_E(L) = \frac{R_0}{\sqrt{1 - e^2 \sin^2(L)}} \qquad (1)$$

- Meridian radius of curvature

$$R_N(L) = \frac{(1 - e^2)R_0}{\left(1 - e^2 \sin^2(L)\right)^{3/2}} \qquad (2)$$





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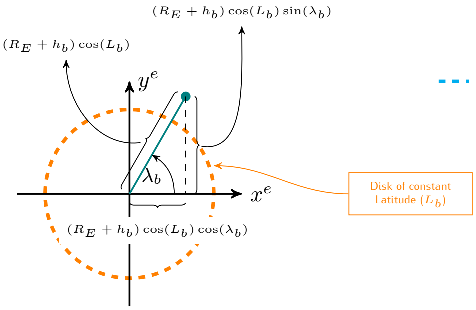
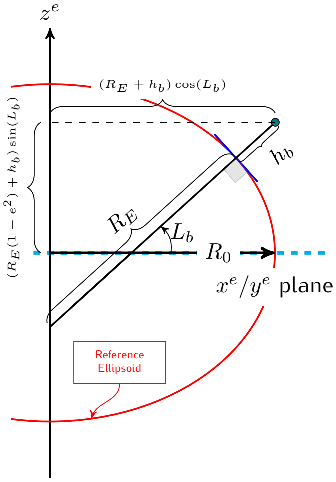
Earth Modeling

$$\vec{r}^{\,e}_{eb} = \begin{bmatrix} x^e_{eb} \\ y^e_{eb} \\ z^e_{eb} \end{bmatrix} = \begin{bmatrix} (R_E + h_b)\cos(L_b)\cos(\lambda_b) \\ (R_E + h_b)\cos(L_b)\sin(\lambda_b) \\ (R_E(1 - e^2) + h_b)\sin(L_b) \end{bmatrix}$$





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Gravity Models

- Specific force ($\vec{f}_{ib}$)
  - Non-gravitational force per unit mass (unit of acceleration)
    - Accelerometers measure specific force
- Specific force sensed when stationary (*wrt* earth) is referred to as the acceleration due to gravity ($\vec{g}_b$)
  - Actually, the reaction to this force
- Gravitational force (γ_{ib}) is result of mass attraction
  - The gravitational mass attraction force is different from the acceleration due to gravity

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Gravity Models

- Relationship between specific force, inertial acceleration, and gravitational attraction

Specific force

$$\vec{f}_{ib} = \vec{a}_{ib} - \vec{\gamma}_{ib} \qquad (3)$$

- When stationary on the surface of the earth
  - A fixed point in a rotating frame

$$\ddot{\vec r}^{\,0}_{02}(t) = \overset{\dot\omega = 0}{\cancel{\dot{\vec\omega}^{\,0}_{01} \times \vec r^{\,0}_{12}(t)}} \qquad + \vec\omega^{\,0}_{01} \times \left(\vec\omega^{\,0}_{01} \times \vec r^{\,0}_{12}(t)\right)$$

∗ Consider frame {0} to be the {i} frame, {1}={e}, and {2}={b} gives

$$\ddot{\vec r}^{\,i}_{ib}(t) = \vec\omega^{\,i}_{ie} \times \left(\vec\omega^{\,i}_{ie} \times \vec r^{\,i}_{eb}(t)\right)$$

∗ coordinatizing in the e-frame

$$\ddot{\vec r}^{\,e}_{ib}(t) = \vec\omega^{\,e}_{ie} \times (\vec\omega^{\,e}_{ie} \times \vec r^{\,e}_{eb}(t))$$

Gravity Models

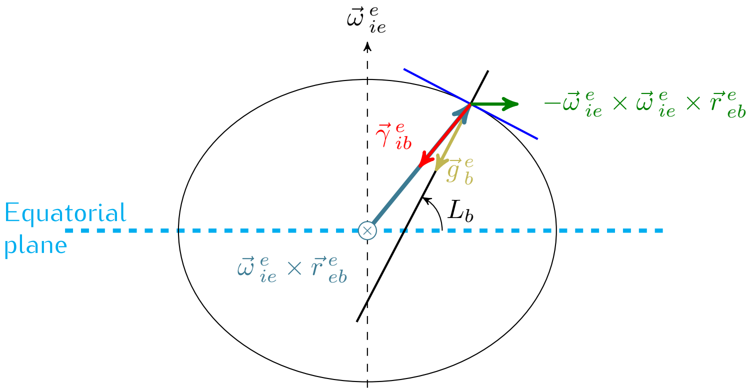
- Thus, when stationary on the surface of the earth the acceleration is due to centrifugal force

$$\vec a^{\,e}_{ib} = \Omega^e_{ie}\Omega^e_{ie}\vec r^{\,e}_{eb}$$

- Therefore, the acceleration due to gravity is

$$\vec g^{\,e}_{b} = -\left.\vec f_{ib}\right|_{\vec v^{\,e}_{eb}=0} = -\Omega^e_{ie}\Omega^e_{ie}\vec r^{\,e}_{eb} + \vec\gamma^{\,e}_{ib} \tag{4}$$

Gravity Models



$$\underbrace{\vec g^{\,e}_{b}}_{1\text{g}} = \underbrace{-\Omega^e_{ie}\Omega^e_{ie}\vec r^{\,e}_{eb}}_{\approx 3\text{mg}} + \underbrace{\vec\gamma^{\,e}_{ib}}_{\approx 1\text{g}}$$

Gravity Models

- Now, $\vec\omega^{\,e}_{ie} = \begin{bmatrix}0\\0\\1\end{bmatrix}\omega_{ie}$ and hence, $\Omega^e_{ie} = \begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\omega_{ie}$, and thus

$$\vec g^{\,e}_{b} = \vec\gamma^{\,e}_{ib} + \omega^2_{ie}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\vec r^{\,e}_{eb}$$

- The WGS 84 model of acceleration due to gravity (on the ellipsoid) can be approximated by (Somigliana model)

$$g_0(L_b) = 9.7803253359\frac{(1 + 0.001931853\sin^2(L))}{\sqrt{1 - e^2\sin^2(L)}} \tag{5}$$

Gravity as a Function of L_b, λ_b and h_b

$$g^n_{b,D} = g_0(L_b, h_b)\left\{1 - \frac{2}{R_0}\left[1 + f(1 - 2\sin^2 L_b) + \frac{\omega_{ie}^2 R_0^2 R_p}{\mu}\right] h_b + \frac{3}{R_0^2} h_b^2\right\} \quad (6)$$

where $\mu = 3.986004418 \times 10^{14}$ m^3/s^2 is the WGS 84 Earth's gravitational constant.

Gravity Models

- On March 17, 2002 NASA launched the Gravity Recovery and Climate Experiment (GRACE) which led to the development of some of the most precise Earth gravity models.