EE 565: Position, Navigation, and Timing

Kalman Filtering Example

Aly El-Osery Kevin Wedeward

Electrical Engineering Department, New Mexico Tech
Socorro, New Mexico, USA

In Collaboration with
Stephen Bruder
Electrical and Computer Engineering Department
Embry-Riddle Aeronautical Univesity
Prescott, Arizona, USA

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$$\dot{\vec{x}}(t) = F(t)\vec{x}(t) + G(t)\vec{w}(t) \tag{1}$$

$$\vec{y}(t) = H(t)\vec{x}(t) + \vec{v}(t) \tag{2}$$

System Discretization

$$\Phi_{k-1} = e^{F_{k-1}\tau_s} \approx \mathcal{I} + F_{k-1}\tau_s \tag{3}$$

where F_{k-1} is the average of F at times t and $t - \tau_s$, and first order approximation is used. Leading to

$$\vec{x}_k = \ \Phi_{k-1}\ \vec{x}_{k-1} + \vec{w}_{k-1} \tag{4}$$

$$\vec{z}_k = \ H_k\ \vec{x}_k + \vec{v}_k \tag{5}$$

- $\vec{w}_k$ and $\vec{v}_k$ are drawn from a Gaussian distribution, uncorrelated have zero mean and statistically independent.

$$\mathbb{E}\{\vec{w}_k \vec{w}_i^T\} = \begin{cases} Q_k & i = k \\ 0 & i \neq k \end{cases} \tag{6}$$

$$\mathbb{E}\{\vec{v}_k \vec{v}_i^T\} = \begin{cases} R_k & i = k \\ 0 & i \neq k \end{cases} \tag{7}$$

$$\mathbb{E}\{\vec{w}_k \vec{v}_i^T\} = \begin{cases} 0 & \forall i, k \end{cases} \tag{8}$$

- State covariance matrix

$$Q_{k-1} \approx \frac{1}{2}\left[\Phi_{k-1} G_{k-1} Q(t_{k-1}) G_{k-1}^T \Phi_{k-1}^T + G_{k-1} Q(t_{k-1}) G_{k-1}^T\right] \tau_s \tag{9}$$

Review: Kalman filter data flow

Initial estimate ($\hat{\vec{x}}_0$ and P_0)

Compute Kalman gain

$$K_k = P_{k|k-1} H_k^T \left(H_k P_{k|k-1} H_k^T + R_k \right)^{-1}$$

Update estimate with measurement $\vec{z}_k$

$$\hat{\vec{x}}_{k|k} = \hat{\vec{x}}_{k|k-1} + K_k \left(\vec{z}_k - H_k \hat{\vec{x}}_{k|k-1} \right)$$

Update error covariance

$$\boldsymbol{P}_{k|k} = \left(\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{H}_k \right) \boldsymbol{P}_{k|k-1} \left(\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{H}_k \right)^T + \boldsymbol{K}_k \boldsymbol{R}_k \boldsymbol{K}_k^T$$

$k = k + 1$

Project ahead

$$\hat{\vec{x}}_{k|k-1} = \Phi_{k-1} \hat{\vec{x}}_{k-1|k-1}$$

$$P_{k|k-1} = Q_{k-1} + \Phi_{k-1} P_{k-1|k-1} \Phi_{k-1}^T$$

Remarks

- Kalman filter (KF) is optimal under the assumptions that the system is linear and the noise is uncorrelated
- Under these assumptions KF provides an unbiased and minimum variance estimate.
- If the Gaussian assumptions is not true, Kalman filter is biased and not minimum variance.
- If the noise is correlated we can augment the states of the system to maintain the uncorrelated requirement of the system noise.

Given a state space system

$$\dot{\vec{x}}_1(t) = F_1(t)\vec{x}_1(t) + G_1(t)\vec{w}_1(t)$$

$$\vec{y}_1(t) = H_1(t)\vec{x}_1(t) + \vec{v}_1(t)$$

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As we have seen the noise $\vec{w}_1(t)$ may be non-white, e.g., correlated Gaussian noise, and as such may be modeled as

$$\dot{\vec{x}}_2(t) = F_2(t)\vec{x}_2(t) + G_2(t)\vec{w}_2(t)$$

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$$\vec{w}_1(t) = H_2(t)\vec{x}_2(t)$$

Define a new augmented state

$$\vec{x}_{aug} = \begin{pmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{pmatrix} \tag{10}$$

therefore,

$$\dot{\vec{x}}_{aug} = \begin{pmatrix} \dot{\vec{x}}_1(t) \\ \dot{\vec{x}}_2(t) \end{pmatrix} = \begin{pmatrix} F_1(t) & G_1 H_2(t) \\ 0 & F_2(t) \end{pmatrix} \begin{pmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ G_2(t) \end{pmatrix} \vec{w}_2(t) \tag{11}$$

and

$$\vec{y}(t) = \begin{pmatrix} H_1(t) & 0 \end{pmatrix} \begin{pmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{pmatrix} + \vec{v}_1(t) \tag{12}$$

Given a state space system

$$\dot{\vec{x}}_1(t) = F_1(t)\vec{x}_1(t) + G_1(t)\vec{w}(t)$$

$$\vec{y}_1(t) = H_1(t)\vec{x}_1(t) + \vec{v}_1(t)$$

In this case the measurement noise $\vec{v}_1$ may be correlated

$$\dot{\vec{x}}_2(t) = F_2(t)\vec{x}_2(t) + G_2(t)\vec{v}_2(t)$$

$$\vec{v}_1(t) = H_2(t)\vec{x}_2(t)$$

Define a new augmented state

$$\vec{x}_{aug} = \begin{pmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{pmatrix} \tag{13}$$

therefore,

$$\dot{\vec{x}}_{aug} = \begin{pmatrix} \dot{\vec{x}}_1(t) \\ \dot{\vec{x}}_2(t) \end{pmatrix} = \begin{pmatrix} F_1(t) & 0 \\ 0 & F_2(t) \end{pmatrix} \begin{pmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{pmatrix} + \begin{pmatrix} G_1(t) & 0 \\ 0 & G_2(t) \end{pmatrix} \begin{pmatrix} \vec{w}(t) \\ \vec{v}_2(t) \end{pmatrix} \tag{14}$$

and

$$\vec{y}(t) = \begin{pmatrix} H_1(1) & H_2(t) \end{pmatrix} \begin{pmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{pmatrix} \tag{15}$$

You are to design a system that estimates the position and velocity of a moving point in a straight line. You have:

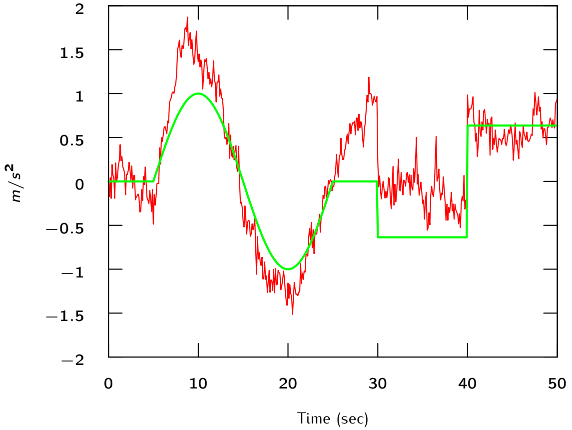
1. an accelerometer corrupted with noise
2. an aiding sensor allowing you to measure absolute position that is also corrupted with noise.

Specification

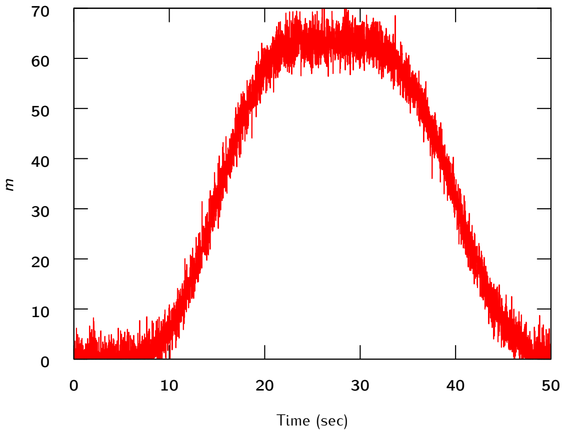
- Sampling Rate Fs = 100Hz.
- Accelerometer specs
  1. VRW $= 1mg/\sqrt{Hz}$.
  2. BI $= 7mg$ with correlation time 6s.
- Position measurement is corrupted with WGN. $\sim \mathcal{N}(0, \sigma_p^2)$, where $\sigma_p = 2.5$m

Input - Acceleration

True Acceleration and Acceleration with Noise

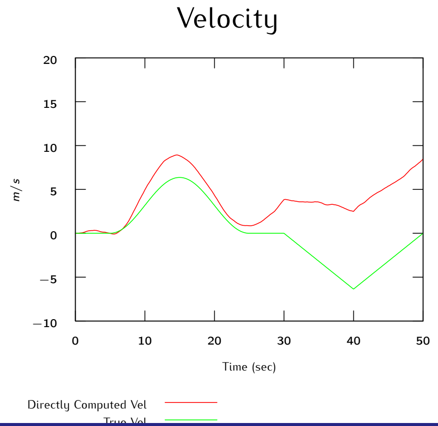


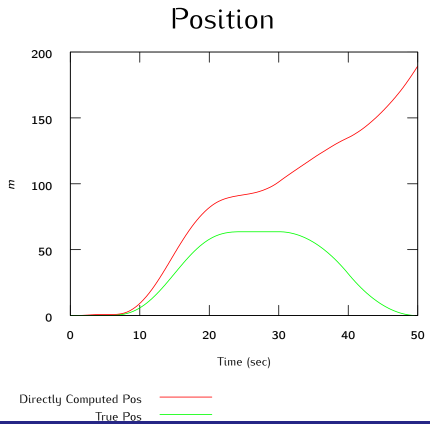
Absolute position measurement corrupted with noise



Computed Position and Velocity

Using only the acceleration measurement and an integration approach to compute the velocity, then integrate again to get position.





1. Clean up the noisy input to the system by filtering
2. Use Kalman filtering techniques with

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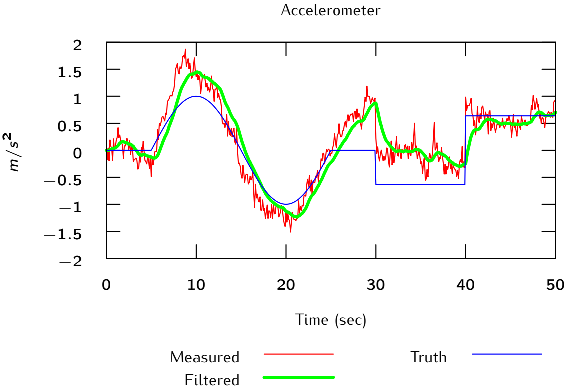
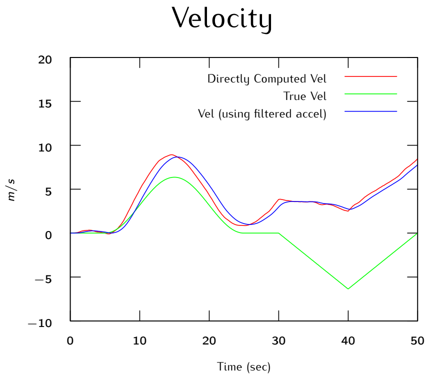
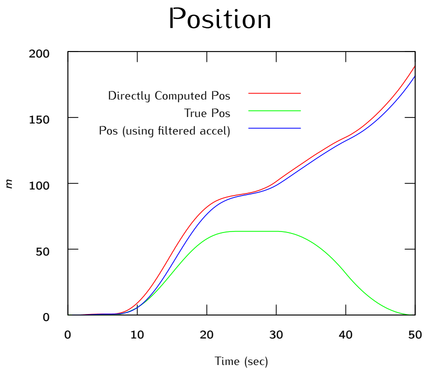
2. Use Kalman filtering techniques with
   - A model of the system dynamics (too restrictive)
   - A model of the error dynamics and correct the system output in
     - open-loop configuration, or
     - closed-loop configuration.

Filtered Accel Measurement



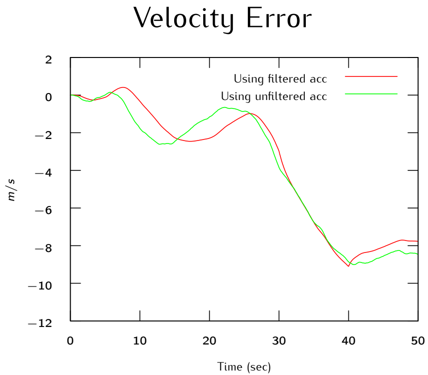


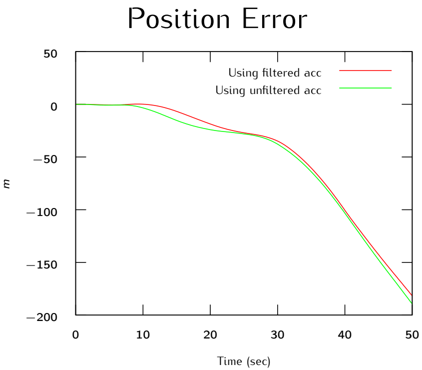
Velocity
Directly Computed Vel
True Vel
Vel (using filtered accel)
m/s
20
15
10
5
0
−5
−10
0
10
20
30
40
50
Time (sec)

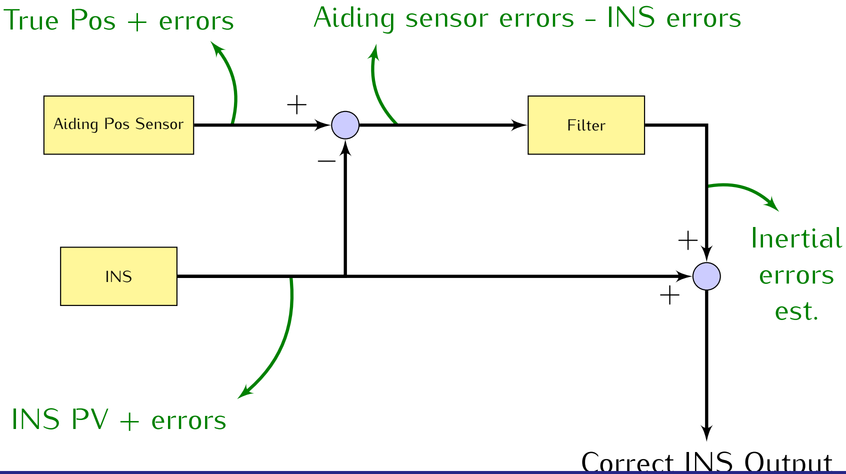


Position
Directly Computed Pos
True Pos
Pos (using filtered accel)
m
200
150
100
50
0
0
10
20
30
40
50
Time (sec)

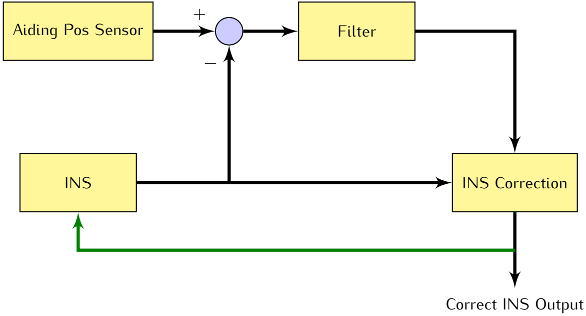
Position and Velocity Errors






True Pos + errors
Aiding sensor errors - INS errors
Aiding Pos Sensor
+
−
Filter
Inertial errors est.
INS
+
+
INS PV + errors
Correct INS Output

If error estimates are fedback to correct the INS mechanization, a reset of the state estimates becomes necessary.



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- State noise covariance matrix (continuous)

$$\mathbb{E}\{\vec{w}(t)\vec{w}^T(\tau)\} = Q(t)\delta(t-\tau)$$

- State noise covariance matrix (discrete)

$$\mathbb{E}\{\vec{w}_k\vec{w}_i^T\} = \begin{cases} Q_k & i = k \\ 0 & i \neq k \end{cases}$$

- Measurement noise covariance matrix

$$\mathbb{E}\{\vec{v}_k\vec{v}_i^T\} = \begin{cases} R_k & i = k \\ 0 & i \neq k \end{cases}$$

- Initial error covariance matrix

$$P_0 = \mathbb{E}\{(\vec{x}_0 - \hat{\vec{x}}_0)(\vec{x}_0 - \hat{\vec{x}}_0)^T\} = \mathbb{E}\{\vec{e}_0\hat{\vec{e}}_0^T\}$$

The position, velocity and acceleration may be modeled using the following kinematic model.

$$\begin{aligned} \dot{p}(t) &= v(t) \\ \dot{v}(t) &= a(t) \end{aligned} \tag{16}$$

where $a(t)$ is the input. Therefore, our estimate of the position is $\hat{p}(t)$ that is the double integration of the acceleration.

Assuming that the accelerometer sensor measurement may be modeled as

$$\tilde{a}(t) = a(t) + b(t) + w_a(t) \tag{17}$$

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and the bias is Markov, therefore

$$\dot{b}(t) = -\frac{1}{T_c} b(t) + w_b(t) \tag{18}$$

where $w_a(t)$ and $w_b(t)$ are zero mean WGN with variances, respectively, $Fs \cdot VRW^2$

$$\mathbb{E}\{w_b(t) w_b(t+\tau)\} = Q_b(t)\delta(t-\tau) \tag{19}$$

$$Q_b(t) = \frac{2\sigma_{BI}^2}{T_c} \tag{20}$$

and T_c is the correlation time and σ_{BI} is the bias instability.

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$$Q_b(t) = \frac{2\sigma_{BI}^2}{T_c} \tag{20}$$

and T_c is the correlation time and σ_{BI} is the bias instability.
Make sure that the *VRW* and σ_{BI} are converted to have SI units.

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Define error terms as

$$\delta p(t) = p(t) - \hat{p}(t), \tag{21}$$

$$\begin{aligned}\delta \dot{p}(t) &= \dot{p}(t) - \dot{\hat{p}}(t) \\ &= v(t) - \hat{v}(t) \\ &= \delta v(t)\end{aligned} \tag{22}$$

and

$$\begin{aligned}\delta \dot{v}(t) &= \dot{v}(t) - \dot{\hat{v}}(t) \\ &= a(t) - \hat{a}(t) \\ &= -b(t) - w_a(t)\end{aligned} \tag{23}$$

where $b(t)$ is modeled as shown in Eq. 18

$$\dot{\vec{x}}(t) = \begin{pmatrix} \delta\dot{p}(t) \\ \delta\dot{v}(t) \\ \dot{b}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -\frac{1}{T_c} \end{pmatrix} \begin{pmatrix} \delta p(t) \\ \delta v(t) \\ b(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ w_a(t) \\ w_b(t) \end{pmatrix} \\ = F(t)\vec{x}(t) + G(t)\vec{w}(t) \tag{24}$$

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- The continuous state noise covariance matrix $Q(t)$ is

$$Q(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & VRW^2 & 0 \\ 0 & 0 & \frac{2\sigma_{BI}^2}{Tc} \end{pmatrix} \tag{25}$$

- The measurement noise covariance matrix is $R = \sigma_p^2$, where σ_p is the standard deviation of the noise of the absolute position sensor.

Now we are ready to start the implementation but first we have to discretize the system.

$$\vec{x}(k+1) = \Phi(k)\vec{x}(k) + \vec{w}_d(k) \tag{26}$$

where

$$\Phi(k) \approx \mathcal{I} + F dt \tag{27}$$

with the measurement equation

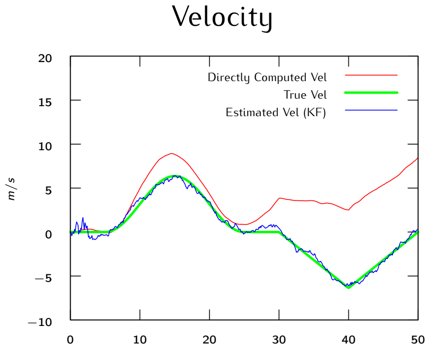
$$y(k) = H\vec{x} + w_p(k) = \delta p(k) + w_p(k) \tag{28}$$

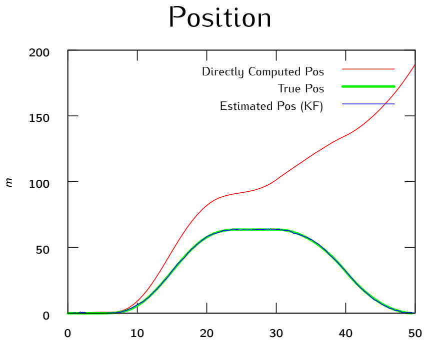
where $H = [1\ 0\ 0]$. The discrete Q_d is approximated as

$$Q_{k-1} \approx \frac{1}{2}[\Phi_{k-1} G(t_{k-1}) Q(t_{k-1}) G^T(t_{k-1}))\Phi_{k-1}^T + G(t_{k-1}) Q(t_{k-1}) G^T(t_{k-1})]dt \tag{29}$$

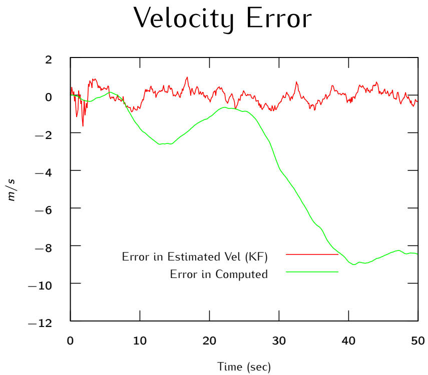
Open-loop Correction

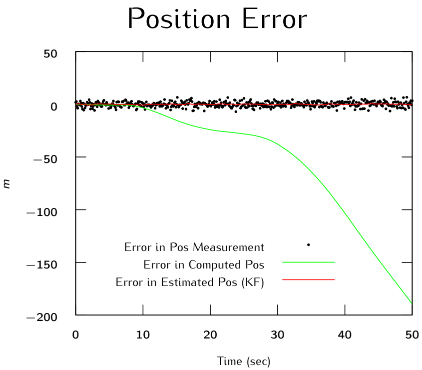
Best estimate = INS out (pos & vel) + KF est error (pos & vel)



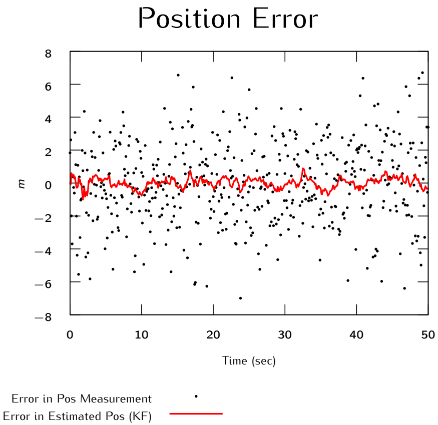


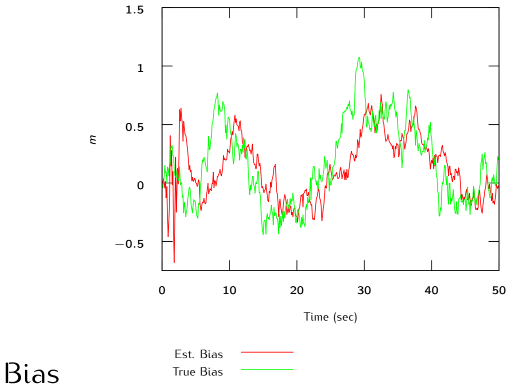
Position and Velocity Errors





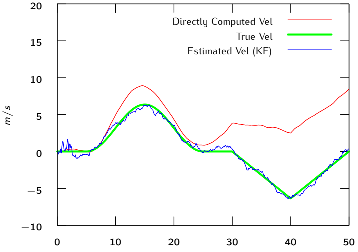
Pos Error & Bias Estimate

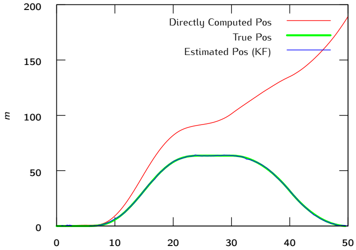


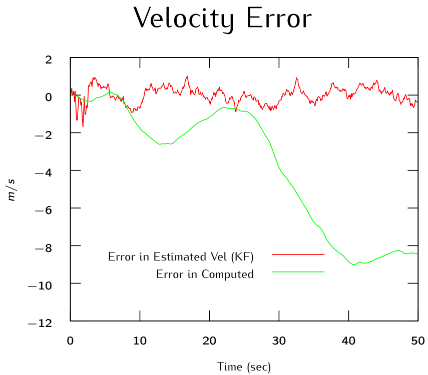


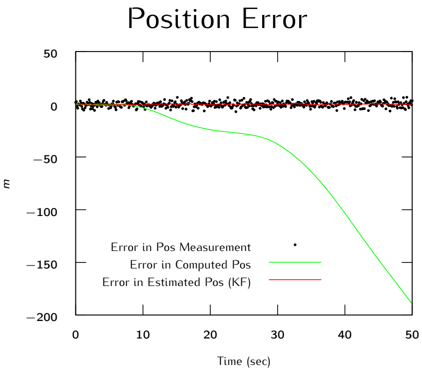
Closed-loop Correction

Best estimate = INS out (pos,vel, & bias) + KF est error (pos, vel & bias)
Use best estimate on next iteration of INS
Accel estimate = accel meas - est bias
Reset state estimates before next call to KF






Velocity Error
2
0
−2
−4
−6
−8
−10
−12
m/s
0
10
20
30
40
50
Time (sec)
Error in Estimated Vel (KF)
Error in Computed


Position Error
50
0
−50
−100
−150
−200
m
0
10
20
30
40
50
Time (sec)
Error in Pos Measurement
Error in Computed Pos
Error in Estimated Pos (KF)

Pos Error & Bias Estimate