EE 565: Position, Navigation and Timing

Navigation Mathematics: Coordinate Frames

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Spring 2023

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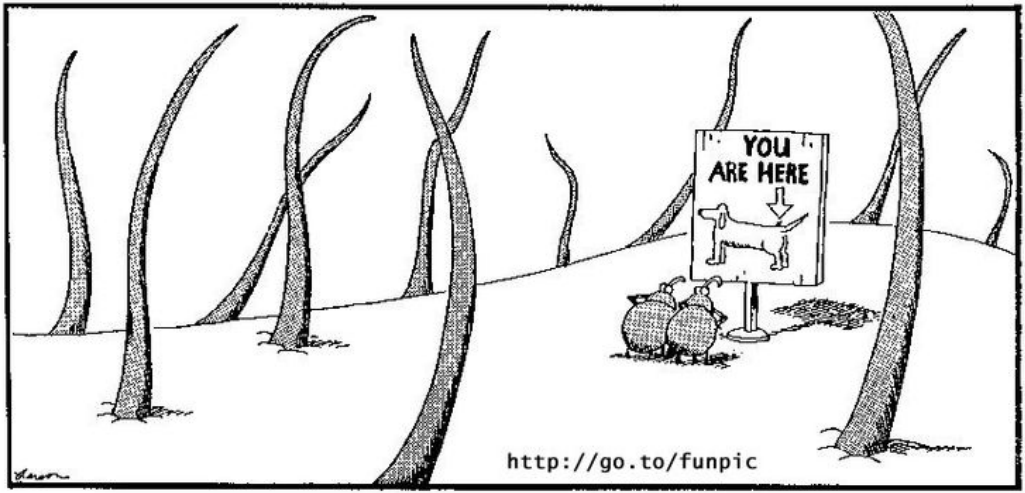
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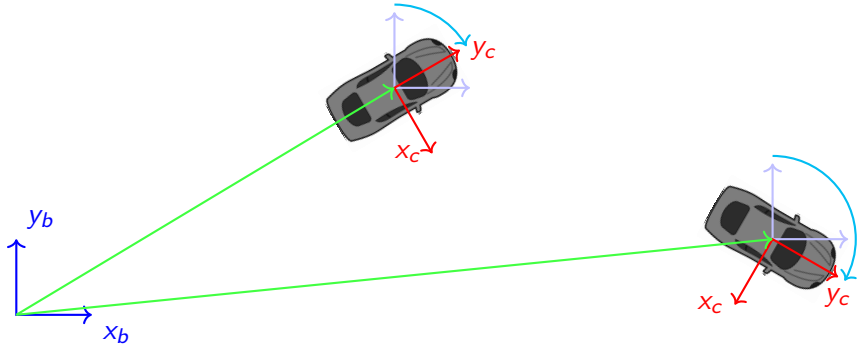
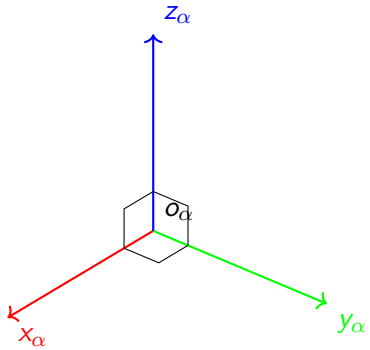
Coordinate Frames

To describe the position and orientation (aka attitude) of objects relative to each other, coordinate frames will be attached and utilized.



Right-hand Cartesian coordinate frame α has

1. origin o_α at which frame is located, and
2. orthonormal basis vectors $x_\alpha, y_\alpha, z_\alpha$ that serve as axes and indicate positive directions.



This definition implies

$$x_\alpha \cdot x_\alpha = y_\alpha \cdot y_\alpha = z_\alpha \cdot z_\alpha = 1$$

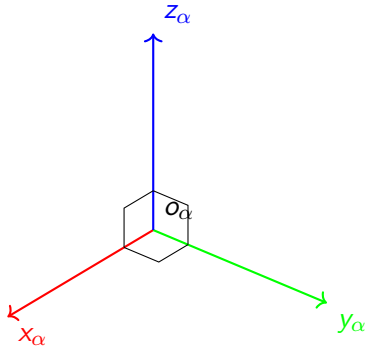
$$x_\alpha \cdot y_\alpha = y_\alpha \cdot z_\alpha = z_\alpha \cdot x_\alpha = 0$$

$$x_\alpha \times y_\alpha = z_\alpha$$

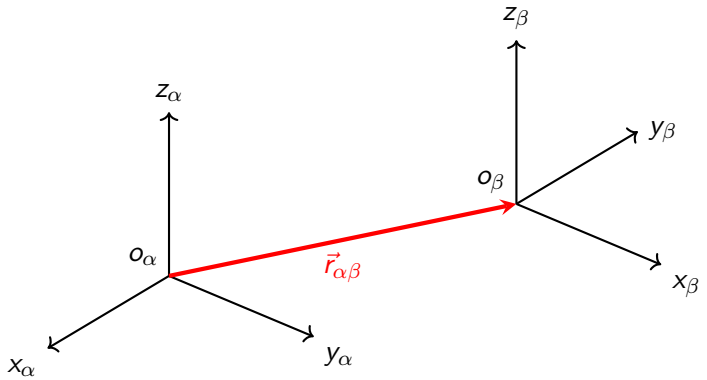
$$y_\alpha \times z_\alpha = x_\alpha$$

$$z_\alpha \times x_\alpha = y_\alpha$$

where '$\cdot$' is the dot (inner) product and '$\times$' is the cross (vector) product.



Coordinate frames are used to enable descriptions of position and orientation/attitude of one frame with respect to another.



Note position is more intuitive than orientation/attitude.

ECI Frame

ECI Frame

- defined as an inertial frame, i.e., it is assumed not to accelerate or rotate with respect to the universe
  - effects of earth's orbit around the sun and motion of the galaxy are very small (smaller than can be measured with inertial sensors), so they will be neglected
  - ECI will be attached to earth and will move with the earth as the earth orbits around the sun, but it won't spin with the earth as it rotates

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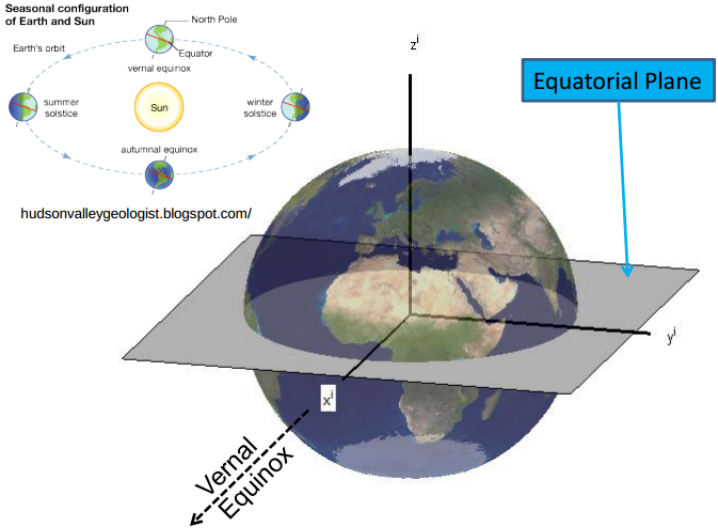
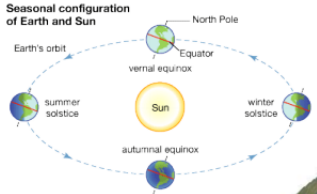
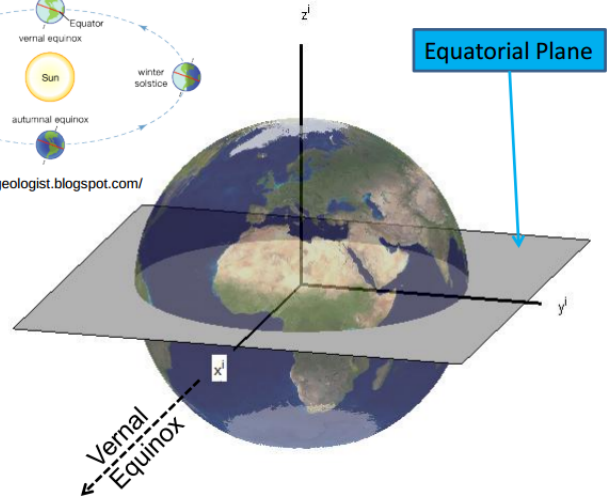
The ECI coordinate frame does **not** rotate with the earth

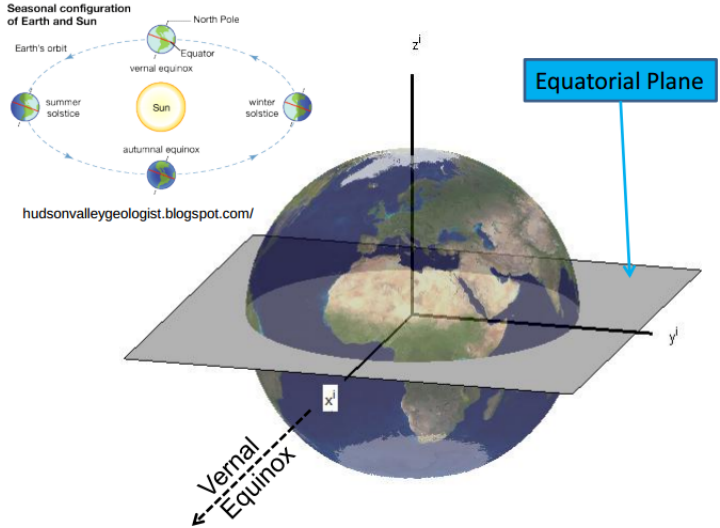

Seasonal configuration of Earth and Sun
North Pole
Equator
Earth's orbit
vernal equinox
summer solstice
Sun
winter solstice
autumnal equinox
hudsonvalleygeologist.blogspot.com/
z^I
Equatorial Plane
y^I
x^I
Vernal Equinox

- o_i at earth's center





- o_i at earth's center
- z_i-axis points along the earth's axis of rotation



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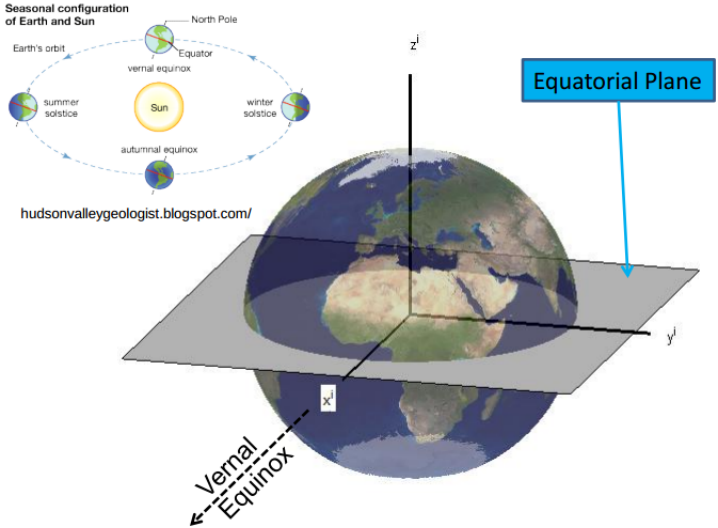

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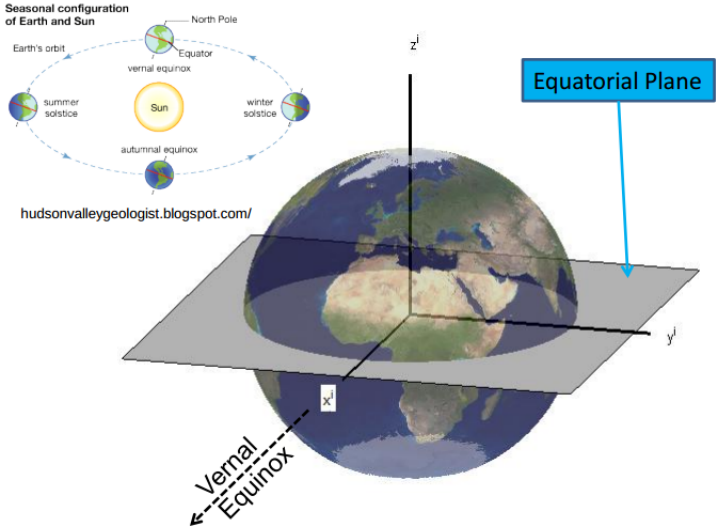

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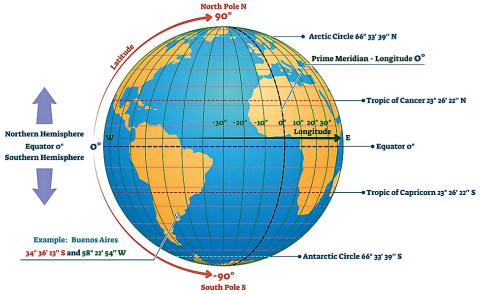

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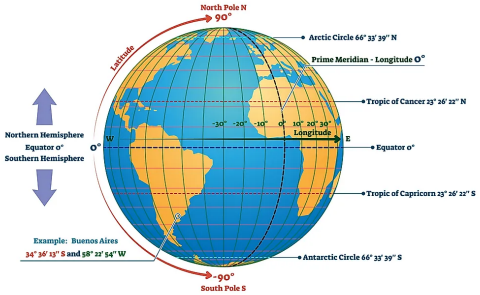

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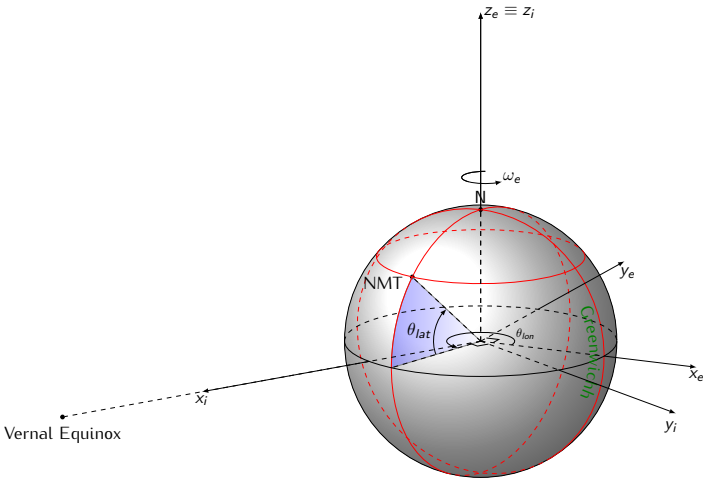
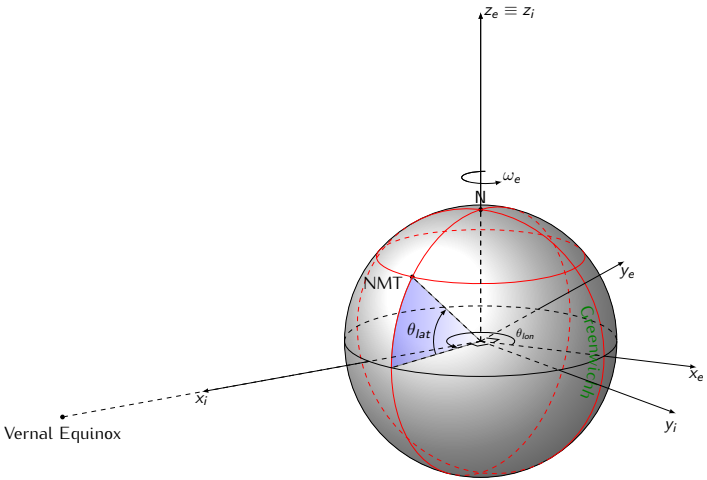

$z_e \equiv z_i$
ω_e
N
NMT
θ_{lat}
θ_{lon}
Greenwich
y_e
x_e
x_i
y_i
Vernal Equinox

- z_e-axis points along axis of earth's rotation



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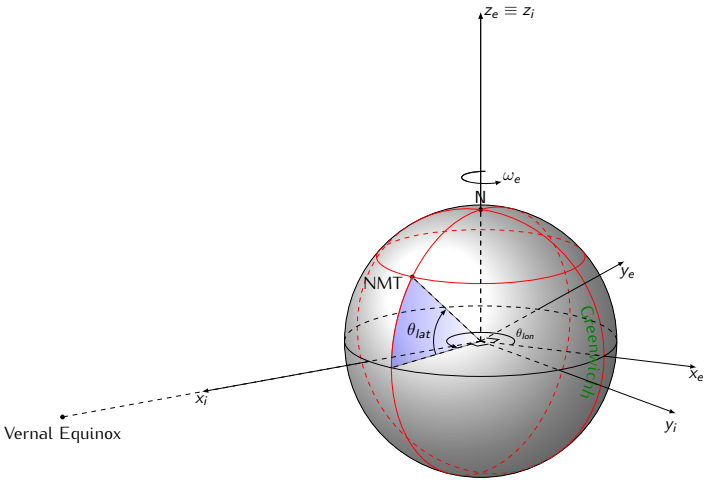
- x_e-axis points towards zero latitude and zero longitude



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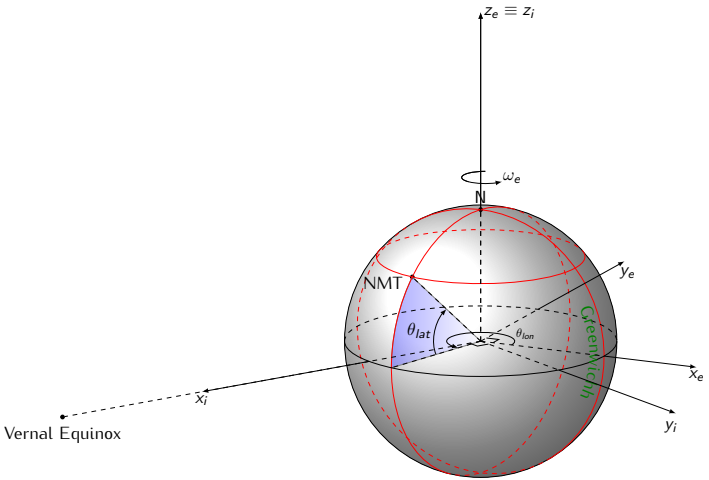

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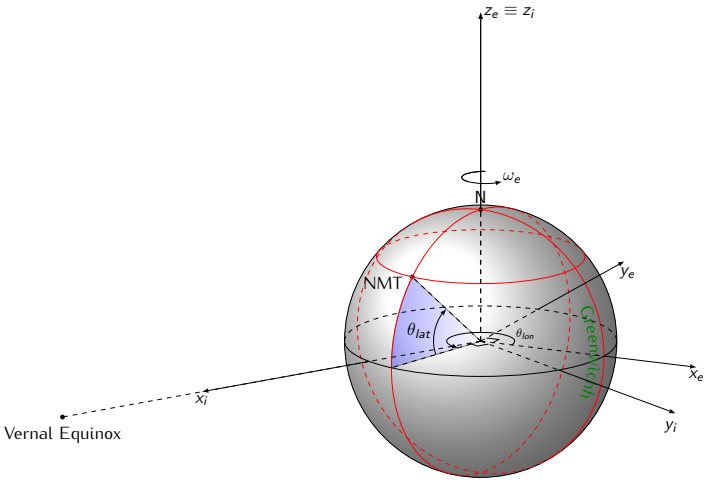
- NMT's (lat, long) $\approx$ $(34.07°, -106.9°) =$ $(34.07°, 253.1°)$



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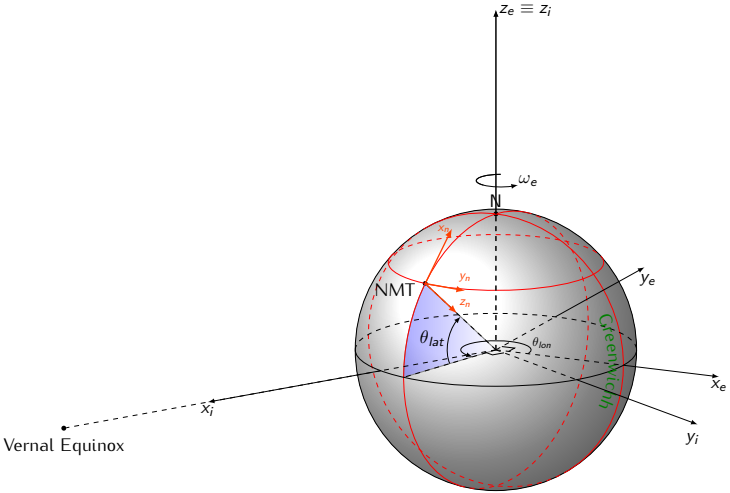
- frame's configuration is often referred to as the NED frame
  - $x_n \rightarrow$ North, $y_n \rightarrow$ East and $z_n \rightarrow$ Down

Nav Frame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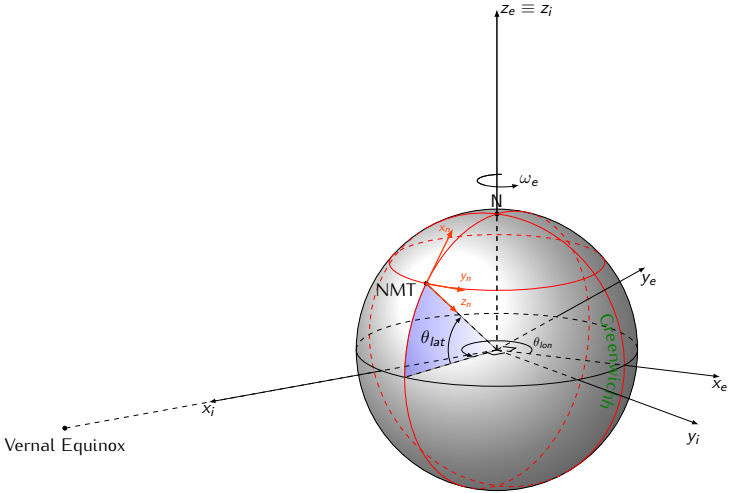
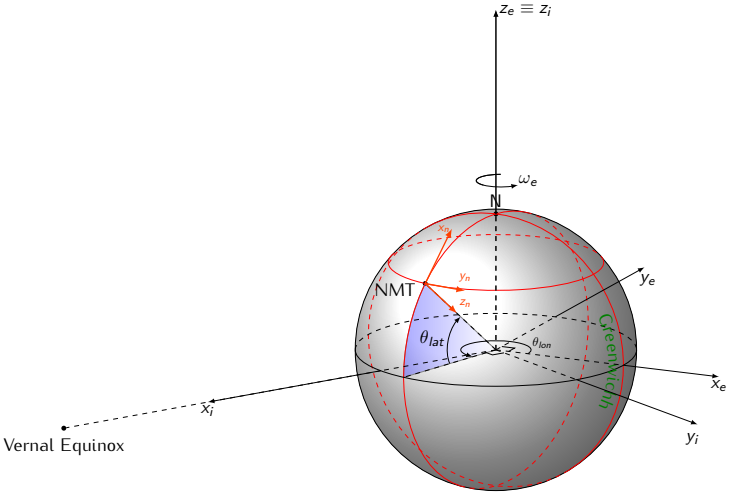
x_i
y_i
Vernal Equinox

- o_n on (potentially moving) body



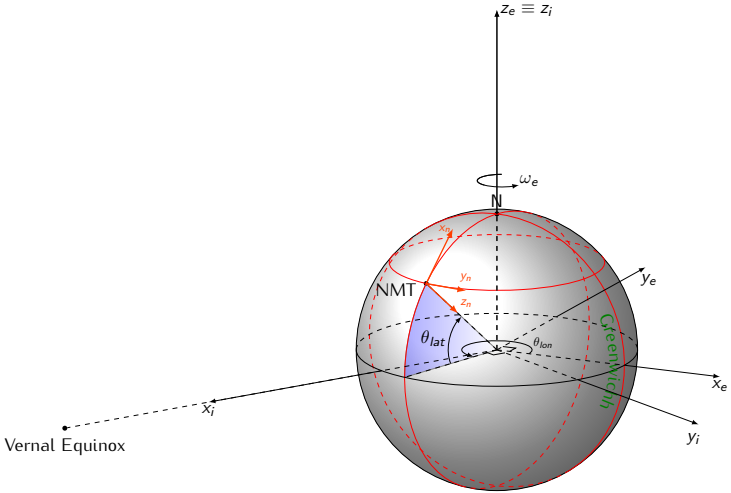
Nav Frame

- o_n on (potentially moving) body
- x_n-axis points north



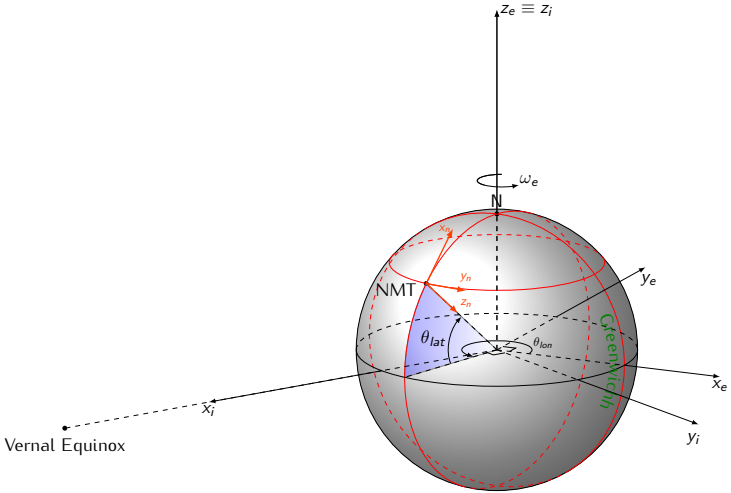
Nav Frame

- o_n on (potentially moving) body
- x_n-axis points north
- y_n-axis points east



Nav Frame

- o_n on (potentially moving) body
- x_n-axis points north
- y_n-axis points east
- z_n-axis points "down"



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  - varies with the roll/pitch of the vehicle

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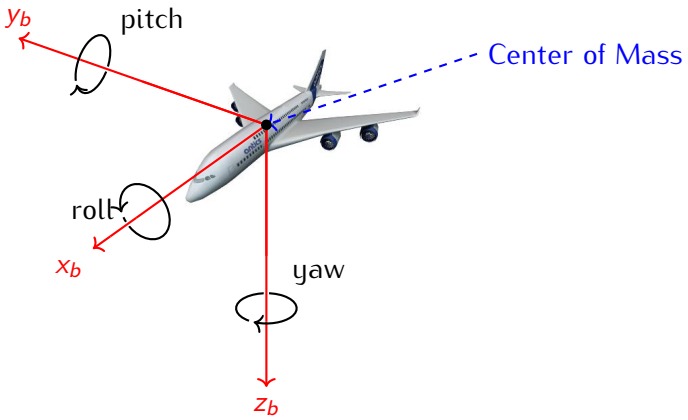
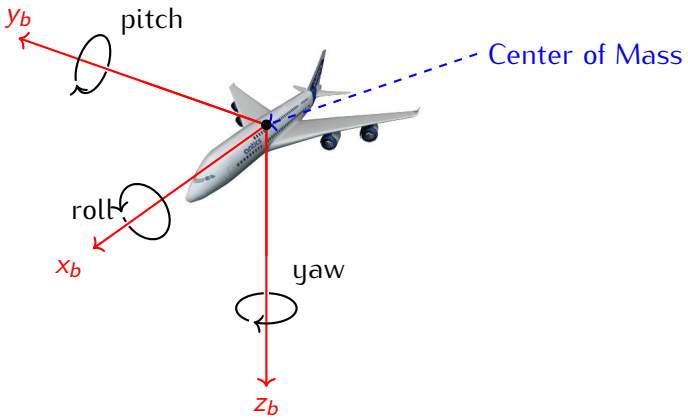
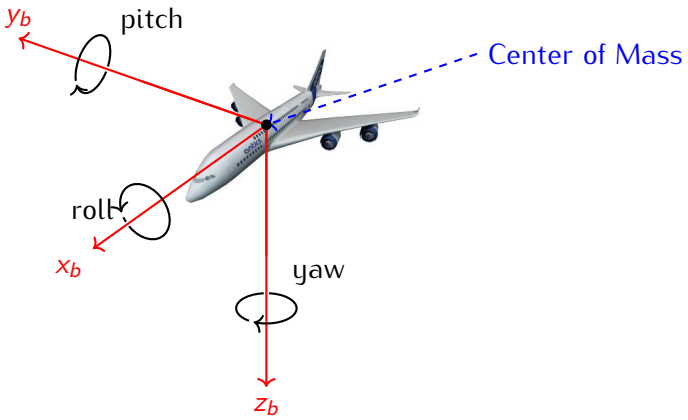
- y_b-axis chosen to complete right hand coordinate system
- referred to as $b-$frame


y_b
pitch
Center of Mass
roll
x_b
yaw
z_b

- body frame is fixed with respect to the vehicle



- body frame is fixed with respect to the vehicle
- x_b "forward"



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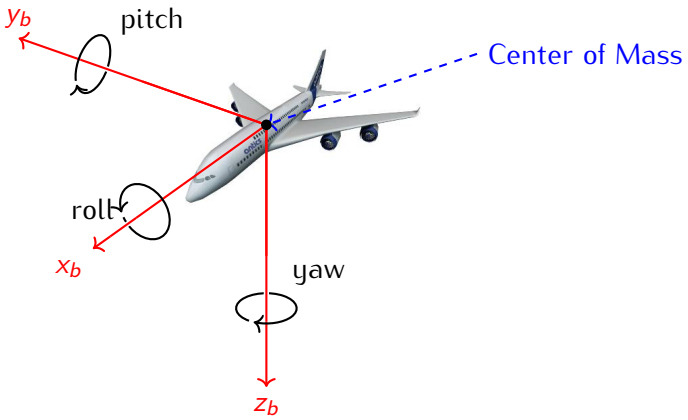
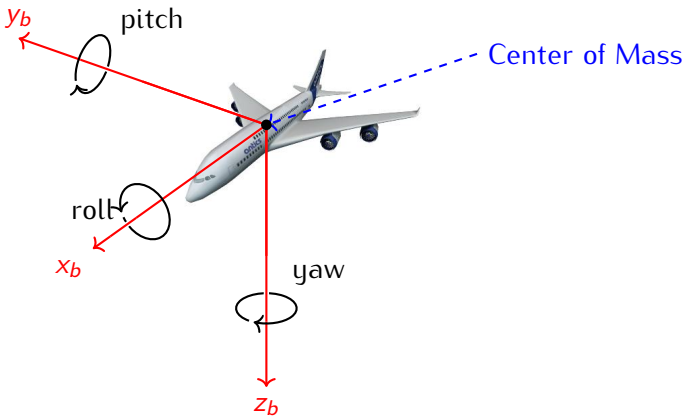
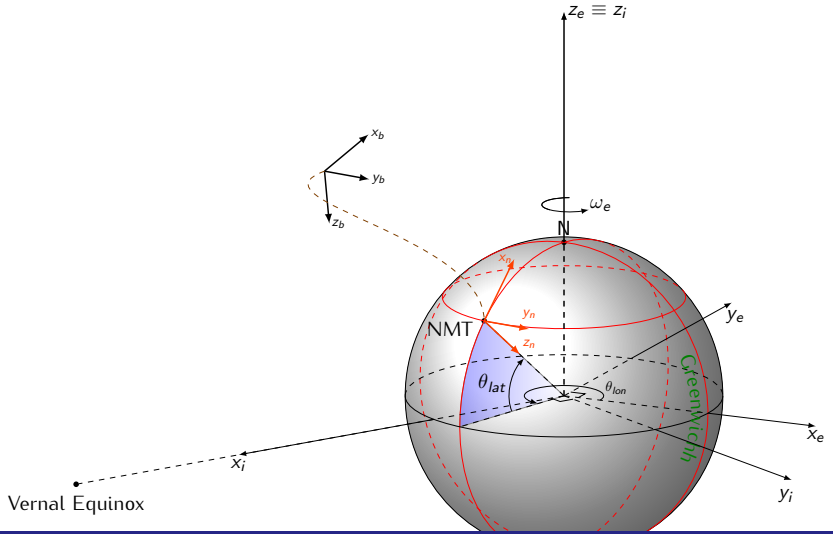


- body frame is fixed with respect to the vehicle
- x_b "forward"
- z_b "down"
- y_b completes right hand coordinate system ("right")

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Other Frames

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- Sensor/Instrument Frame
  - attached to body of sensor that may be displaced from a vehicle's center of mass

The End