

# Lecture

## Navigation Mathematics: Rotation Matrices

EE 565: Position, Navigation and Timing

Lecture Notes Update on Spring 2023

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Lecture Topics

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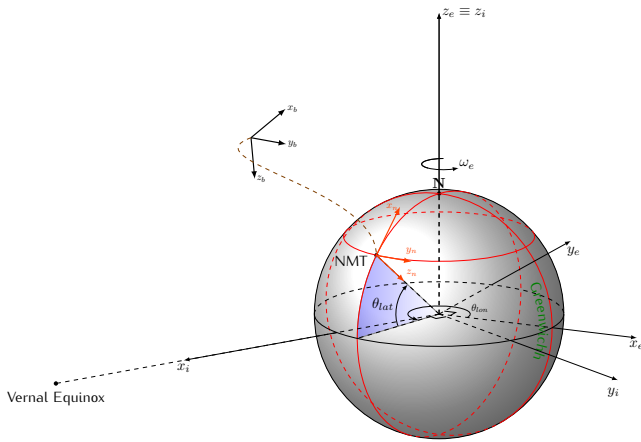
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### 1 Review

Review

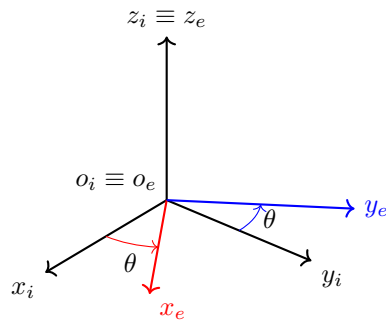
- Coordinate Frames - subscript will "name" axes (vectors)
- Earth-Centered Inertial (ECI) Frame -  $i$
- Earth-Centered Earth-Fixed (ECEF) Frame -  $e$
- Navigation (Nav) Frame -  $n$
- Body Frame -  $b$



## 2 Attitude (Orientation)

### Attitude (Orientation)

- Attitude describes orientation of one coordinate frame with respect to another.
- How would one describe orientation of ECEF frame wrt ECI frame at point in time when angular difference is  $\theta$ ?



- $e$ -frame rotated away from  $i$ -frame by angle  $\theta$  about  $z_i \equiv z_e$

### Attitude (Orientation)

- Less obvious, but equally valid, way of describing  $e$ -frame wrt  $i$ -frame is by giving coordinates of the  $e$ -frame's axes in the  $i$ -frame.
- Leads to need for further notation:

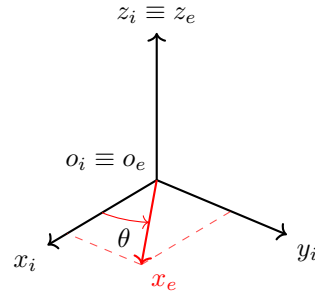
$$- x_e^i \text{ is } x_e = x_e^e = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ coordinatized (written wrt) the } i\text{-frame}$$

$$- y_e^i \text{ is } y_e = y_e^e = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ coordinatized (written wrt) the } i\text{-frame}$$

$$- z_e^i \text{ is } z_e = z_e^e = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ coordinatized (written wrt) the } i\text{-frame}$$

### Attitude (Orientation)

- $x_e^i$ :

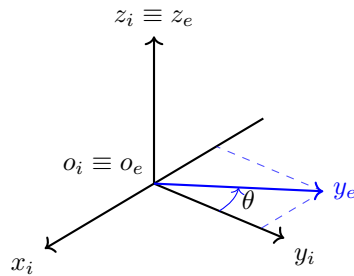


$$\bullet x_e^i = \begin{bmatrix} x_e \cdot x_i \\ x_e \cdot y_i \\ x_e \cdot z_i \end{bmatrix} = \begin{bmatrix} \|x_e\| \|x_i\| \cos(\theta) \\ \|x_e\| \|y_i\| \cos(90^\circ - \theta) \\ \|x_e\| \|z_i\| \cos(90^\circ) \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix}$$

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### Attitude (Orientation)

- $y_e^i$ :

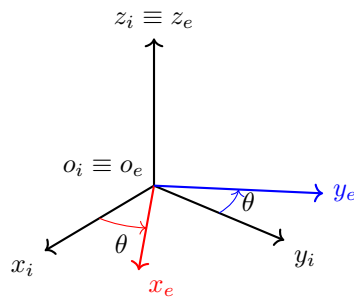


$$\bullet y_e^i = \begin{bmatrix} y_e \cdot x_i \\ y_e \cdot y_i \\ y_e \cdot z_i \end{bmatrix} = \begin{bmatrix} \|y_e\| \|x_i\| \cos(90^\circ + \theta) \\ \|y_e\| \|y_i\| \cos(\theta) \\ \|y_e\| \|z_i\| \cos(90^\circ) \end{bmatrix} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix}$$

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### Attitude (Orientation)

- $z_e^i$ :



$$\bullet z_e^i = \begin{bmatrix} z_e \cdot x_i \\ z_e \cdot y_i \\ z_e \cdot z_i \end{bmatrix} = \begin{bmatrix} \|z_e\| \|x_i\| \cos(90^\circ) \\ \|z_e\| \|y_i\| \cos(90^\circ) \\ \|z_e\| \|z_i\| \cos(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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## Attitude (Orientation)

- $3 \times 3$  matrix can be constructed by using each basis vector of the  $e$ -frame wrt  $i$ -frame as a column

$$\bullet C_e^i = \left[ \begin{array}{c|c|c} x_e^i & y_e^i & z_e^i \end{array} \right] = \left[ \begin{array}{ccc} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{array} \right]$$

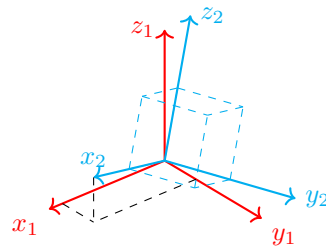
- $C_e^i$  describes the attitude/orientation of the  $e$ -frame wrt the  $i$ -frame
- $C_e^i$  referred to as a rotation matrix, coordinate transformation matrix, or direct cosine matrix (DCM)

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## 3 Rotation Matrices

### Rotation Matrices

- In general, a rotation matrix  $C_2^1$  describes the orientation of **frame {2}** relative to **frame {1}**



$$\bullet \text{ via } C_2^1 = [x_2^1, y_2^1, z_2^1] = \begin{bmatrix} x_2 \cdot x_1 & y_2 \cdot x_1 & z_2 \cdot x_1 \\ x_2 \cdot y_1 & y_2 \cdot y_1 & z_2 \cdot y_1 \\ x_2 \cdot z_1 & y_2 \cdot z_1 & z_2 \cdot z_1 \end{bmatrix}$$

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### Properties of Rotation Matrix

$$\bullet C_2^1 = [x_2^1, y_2^1, z_2^1] = \begin{bmatrix} x_2 \cdot x_1 & y_2 \cdot x_1 & z_2 \cdot x_1 \\ x_2 \cdot y_1 & y_2 \cdot y_1 & z_2 \cdot y_1 \\ x_2 \cdot z_1 & y_2 \cdot z_1 & z_2 \cdot z_1 \end{bmatrix} = \begin{bmatrix} x_1 \cdot x_2 & x_1 \cdot y_2 & x_1 \cdot z_2 \\ y_1 \cdot x_2 & y_1 \cdot y_2 & y_1 \cdot z_2 \\ z_1 \cdot x_2 & z_1 \cdot y_2 & z_1 \cdot z_2 \end{bmatrix} =$$

$$\begin{bmatrix} (x_1^2)^T \\ (y_1^2)^T \\ (z_1^2)^T \end{bmatrix} = [x_1^2, y_1^2, z_1^2]^T = [C_1^2]^T$$

- opposite perspective (frame 2 wrt frame 1 given frame 1 wrt frame 2) is as simple as a matrix transpose!

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### Properties of Rotation Matrix

1.  $[C_2^1]^T C_2^1 = C_1^2 C_2^1 = I \Rightarrow C_1^2 = [C_2^1]^T = [C_2^1]^{-1}$
2.  $|([C_2^1]^T C_2^1)| = |C_2^1| |C_2^1| = |I| \Rightarrow |C_2^1| = \pm 1$  (+ for right hand coordinate system)
3. columns and rows of  $C_2^1$  are orthogonal
4. magnitude of columns and rows in  $C_2^1$  are 1

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## Rotation Matrix as Coordinate Transformation

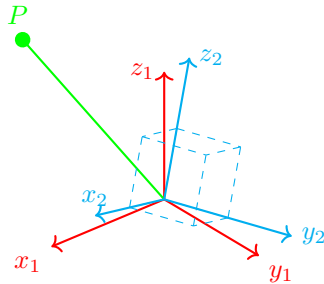
- So far, rotation matrix  $C$  developed to describe orientation
- $C$  can also perform change of coordinates on vector

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## Rotation Matrix as Coordinate Transformation

- Consider a point  $P$  with location described as a vector in coordinate frame {1}

$$\vec{P}^1 = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = ux_1 + vy_1 + wz_1$$



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## Rotation Matrix as Coordinate Transformation

- With  $\vec{P}^1$  given, the location of point  $P$  can be described in coordinate frame {2} via

$$\begin{aligned} \vec{P}^2 &= \begin{bmatrix} \vec{P}^1 \cdot x_2 \\ \vec{P}^1 \cdot y_2 \\ \vec{P}^1 \cdot z_2 \end{bmatrix} = \begin{bmatrix} (ux_1 + vy_1 + wz_1) \cdot x_2 \\ (ux_1 + vy_1 + wz_1) \cdot y_2 \\ (ux_1 + vy_1 + wz_1) \cdot z_2 \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} x_1 \cdot x_2 & y_1 \cdot x_2 & z_1 \cdot x_2 \\ x_1 \cdot y_2 & y_1 \cdot y_2 & z_1 \cdot y_2 \\ x_1 \cdot z_2 & y_1 \cdot z_2 & z_1 \cdot z_2 \end{bmatrix}}_{?} \underbrace{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}_{?} \end{aligned}$$

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## Rotation Matrix as Coordinate Transformation

$$\begin{aligned} &= \underbrace{\begin{bmatrix} x_1 \cdot x_2 & y_1 \cdot x_2 & z_1 \cdot x_2 \\ x_1 \cdot y_2 & y_1 \cdot y_2 & z_1 \cdot y_2 \\ x_1 \cdot z_2 & y_1 \cdot z_2 & z_1 \cdot z_2 \end{bmatrix}}_{C_1^2} \underbrace{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}_{\vec{P}^1} \\ &= C_1^2 \vec{P}^1 \end{aligned}$$

- $\Rightarrow \vec{P}^2 = C_1^2 \vec{P}^1$
- $C_1^2$  re-coordinatized vector written wrt frame 1 into frame 2 by a matrix-multiplication
- superscripts and subscripts help track/denote re-coordinatization

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## Rotation Matrix as Coordinate Transformation

Similarly, coordinate transformations can be performed opposite way as well

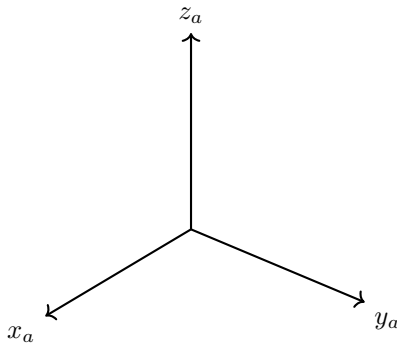
$$\begin{aligned}\vec{P}^2 &= C_1^2 \vec{P}^1 \\ \Rightarrow \vec{P}^1 &= [C_1^2]^{-1} \vec{P}^2 \\ &= [C_1^2]^T \vec{P}^2 \\ &= C_2^1 \vec{P}^2\end{aligned}$$

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## 4 Examples

### Example 1

Given  $C_b^a = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -1 & 0 & 0 \end{bmatrix}$  and frame  $a$ , sketch frame  $b$ .



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### Example 2

Frame 1 has been rotated away from frame 0 by  $30^\circ$  about  $z_0$ . Find  $\vec{r}^0$  given  $\vec{r}^1 = [0, 2, 0]^T$ ,  $\cos(30^\circ) = \frac{\sqrt{3}}{2}$  and  $\sin(30^\circ) = \frac{1}{2}$ .

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## 5 Summary

### Summary

Rotation matrix can be thought of in three distinct ways:

1. It describes the orientation of one coordinate frame *wrt* another coordinate frame
2. It represents a coordinate transformation that relates the coordinates of a point (e.g.,  $P$ ) or vector in two different frames of reference
3. It is an operator that takes a vector  $\vec{p}$  and rotates it into a new vector  $C\vec{p}$ , both in the same coordinate frame

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The End

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