EE 565: Position, Navigation and Timing

Navigation Mathematics: Rotation Matrices

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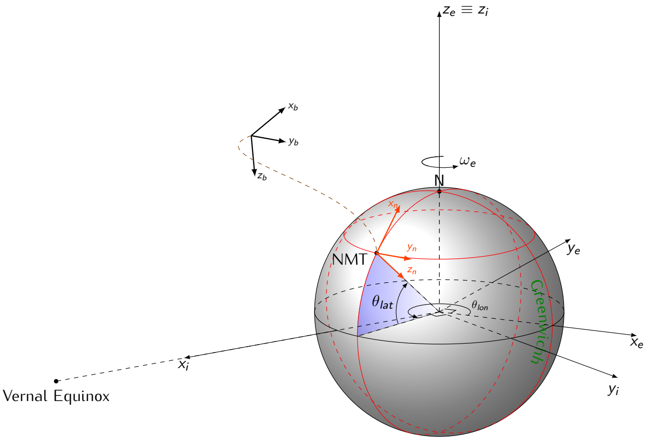
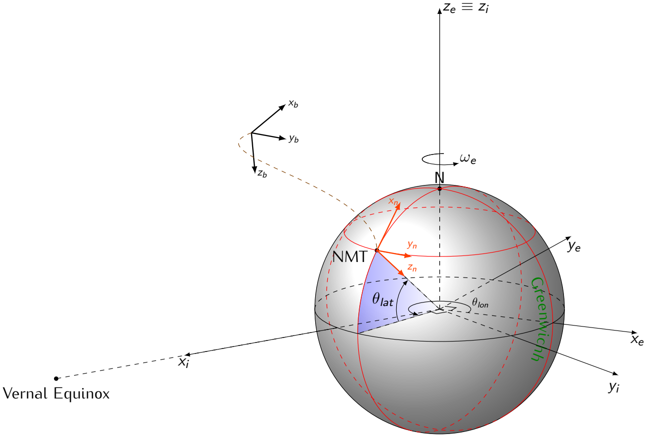
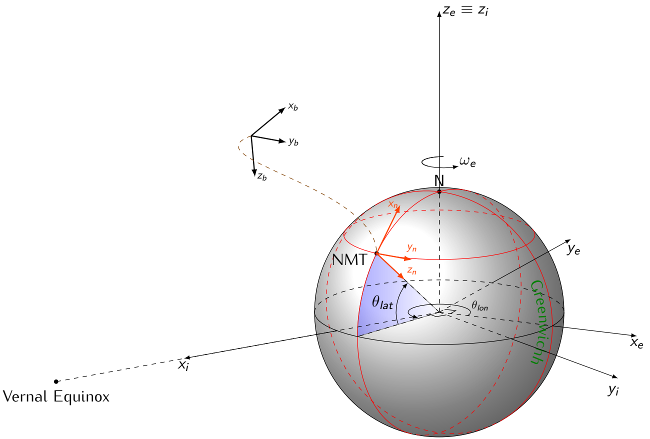
Review


$z_e \equiv z_i$
x_b
y_b
z_b
ω_e
N
x_n
y_n
z_n
y_e
NMT
θ_{lat}
θ_{lon}
Greenwich
x_e
x_i
y_i
Vernal Equinox

- Coordinate Frames – subscript will "name" axes (vectors)



- Coordinate Frames – subscript will "name" axes (vectors)
- Earth-Centered Inertial (ECI) Frame – i



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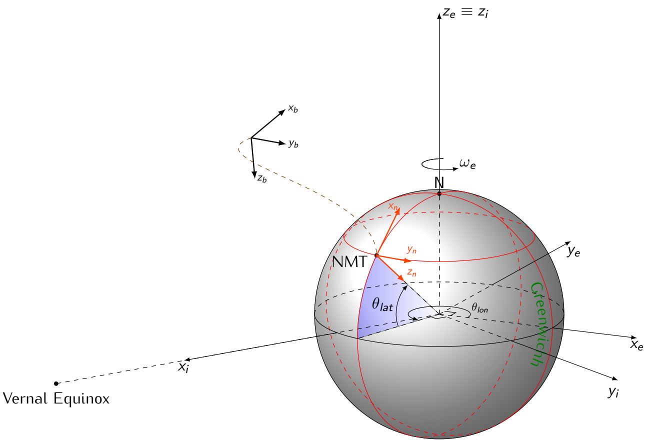

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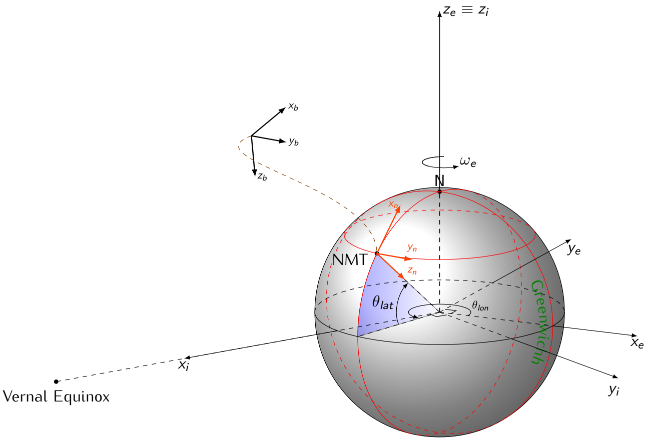

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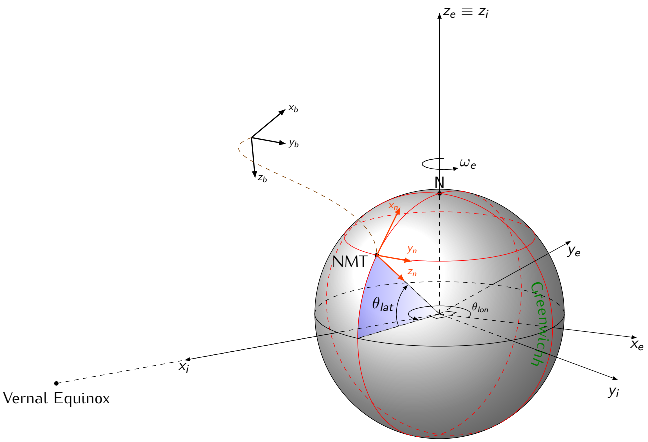
- Earth-Centered Earth-Fixed (ECEF) Frame – e
- Navigation (Nav) Frame – n
- Body Frame – b



Attitude

Attitude (Orientation)

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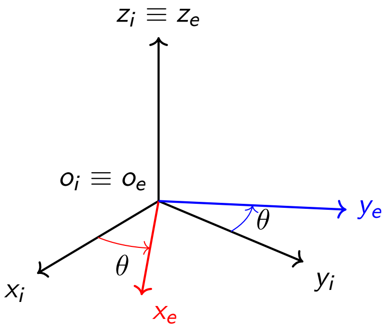

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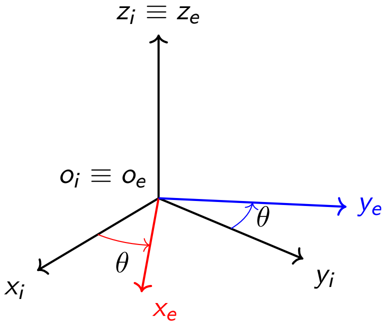

- $e-$frame rotated away from $i-$frame by angle θ about $z_i \equiv z_e$

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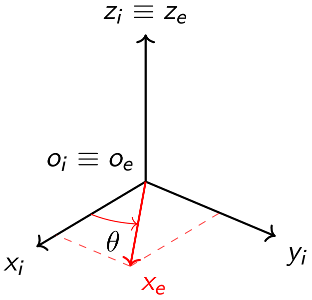

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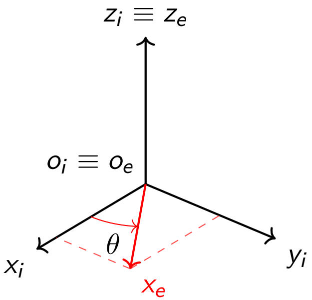
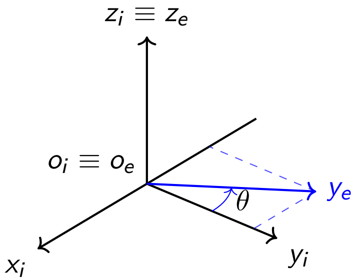


- $x_e^i = \begin{bmatrix} x_e \cdot x_i \\ x_e \cdot y_i \\ x_e \cdot z_i \end{bmatrix} = \begin{bmatrix} \|x_e\|\|x_i\| \cos(\theta) \\ \|x_e\|\|y_i\| \cos(90^\circ - \theta) \\ \|x_e\|\|z_i\| \cos(90^\circ) \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix}$

- y_e^i:



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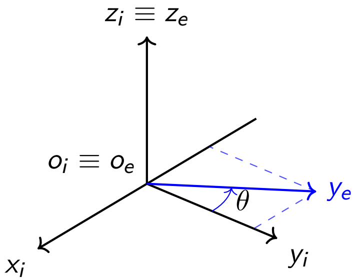
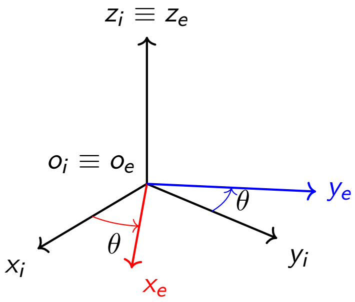


- $$y_e^i = \begin{bmatrix} y_e \cdot x_i \\ y_e \cdot y_i \\ y_e \cdot z_i \end{bmatrix} = \begin{bmatrix} \|y_e\| \|x_i\| \cos(90^\circ + \theta) \\ \|y_e\| \|y_i\| \cos(\theta) \\ \|y_e\| \|z_i\| \cos(90^\circ) \end{bmatrix} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix}$$

- z_e^i:



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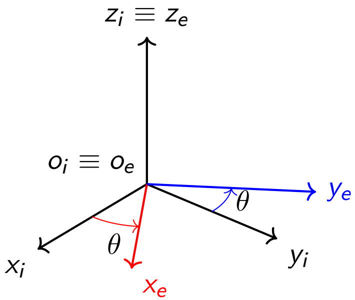

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- $C_e^i = \left[\begin{array}{c|c|c} x_e^i & y_e^i & z_e^i \end{array}\right] = \left[\begin{array}{c|c|c} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{array}\right]$

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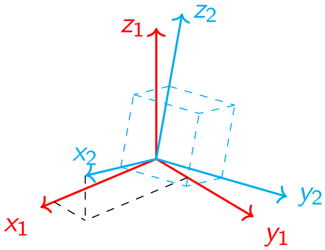
- C_e^i describes the attitude/orientation of the $e-$frame wrt the $i-$frame
- C_e^i referred to as a rotation matrix, coordinate transformation matrix, or direct cosine matrix (DCM)

Rotation Matrices

- In general, a rotation matrix C_2^1 describes the orientation of frame {2} relative to frame {1}



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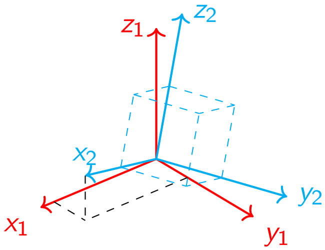

- via $C_2^1 = \left[x_2^1,\ y_2^1,\ z_2^1\right] = \begin{bmatrix} x_2 \cdot x_1, & y_2 \cdot x_1, & z_2 \cdot x_1 \\ x_2 \cdot y_1, & y_2 \cdot y_1, & z_2 \cdot y_1 \\ x_2 \cdot z_1, & y_2 \cdot z_1, & z_2 \cdot z_1 \end{bmatrix}$

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$$= \begin{bmatrix} (x_1^2)^T \\ (y_1^2)^T \\ (z_1^2)^T \end{bmatrix} = [x_1^2,\ y_1^2,\ z_1^2]^T = [C_1^2]^T$$

Properties of Rotation Matrix

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  $$= \begin{bmatrix} (x_1^2)^T \\ (y_1^2)^T \\ (z_1^2)^T \end{bmatrix} = \left[x_1^2,\ y_1^2,\ z_1^2\right]^T = \left[C_1^2\right]^T$$
- opposite perspective (frame 2 wrt frame 1 given frame 1 wrt frame 2) is as simple as a matrix transpose!

Properties of Rotation Matrix

1. $\left[C_2^1\right]^T C_2^1 = C_1^2 C_2^1 = I \;\Rightarrow C_1^2 = \left[C_2^1\right]^T = \left[C_2^1\right]^{-1}$

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4. magnitude of columns and rows in C_2^1 are 1

- So far, rotation matrix C developed to describe orientation

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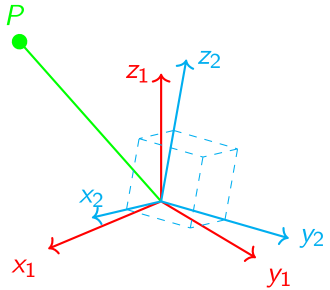
- C can also perform change of coordinates on vector

- Consider a point P with location described as a vector in coordinate frame {1}

$$\vec{P}^{\,1} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = ux_1 + vy_1 + wz_1$$



- With $\vec{P}^{\,1}$ given, the location of point P can be described in coordinate frame $\{2\}$ via

$$\vec{P}^{\,2} = \begin{bmatrix} \vec{P}^{\,1} \cdot x_2 \\ \vec{P}^{\,1} \cdot y_2 \\ \vec{P}^{\,1} \cdot z_2 \end{bmatrix} = \begin{bmatrix} (ux_1 + vy_1 + wz_1) \cdot x_2 \\ (ux_1 + vy_1 + wz_1) \cdot y_2 \\ (ux_1 + vy_1 + wz_1) \cdot z_2 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} x_1 \cdot x_2 & y_1 \cdot x_2 & z_1 \cdot x_2 \\ x_1 \cdot y_2 & y_1 \cdot y_2 & z_1 \cdot y_2 \\ x_1 \cdot z_2 & y_1 \cdot z_2 & z_1 \cdot z_2 \end{bmatrix}}_{?} \underbrace{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}_{?}$$

$$= \underbrace{\begin{bmatrix} x_1 \cdot x_2 & y_1 \cdot x_2 & z_1 \cdot x_2 \\ x_1 \cdot y_2 & y_1 \cdot y_2 & z_1 \cdot y_2 \\ x_1 \cdot z_2 & y_1 \cdot z_2 & z_1 \cdot z_2 \end{bmatrix}}_{C_1^2} \underbrace{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}_{\vec{P}^1}$$

$$= C_1^2 \vec{P}^1$$

- $\Rightarrow \vec{P}^2 = C_1^2 \vec{P}^1$

$$= \underbrace{\begin{bmatrix} x_1 \cdot x_2 & y_1 \cdot x_2 & z_1 \cdot x_2 \\ x_1 \cdot y_2 & y_1 \cdot y_2 & z_1 \cdot y_2 \\ x_1 \cdot z_2 & y_1 \cdot z_2 & z_1 \cdot z_2 \end{bmatrix}}_{C_1^2} \underbrace{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}_{\vec{P}^1}$$

$$= C_1^2 \vec{P}^1$$

- $\Rightarrow \vec{P}^2 = C_1^2 \vec{P}^1$
- C_1^2 re-coordinatized vector written wrt frame 1 into frame 2 by a matrix-multiplication

$$= \underbrace{\begin{bmatrix} x_1 \cdot x_2 & y_1 \cdot x_2 & z_1 \cdot x_2 \\ x_1 \cdot y_2 & y_1 \cdot y_2 & z_1 \cdot y_2 \\ x_1 \cdot z_2 & y_1 \cdot z_2 & z_1 \cdot z_2 \end{bmatrix}}_{C_1^2} \underbrace{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}_{\vec{P}^{\,1}}$$

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- $\Rightarrow \vec{P}^{\,2} = C_1^2 \vec{P}^{\,1}$
- C_1^2 re-coordinatized vector written wrt frame 1 into frame 2 by a matrix-multiplication
- superscripts and subscripts help track/denote re-coordinatization

Similarly, coordinate transformations can be performed opposite way as well

$$\vec{P}^{\,2} = C_1^2 \vec{P}^{\,1}$$

$$\Rightarrow \vec{P}^{\,1} = [C_1^2]^{-1} \vec{P}^{\,2}$$

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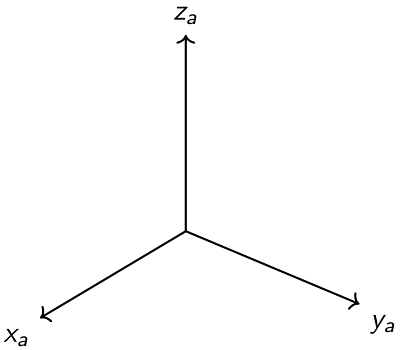
$$= [C_1^2]^T \vec{P}^{\,2}$$

$$= C_2^1 \vec{P}^{\,2}$$

Examples

Given $C_b^a = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -1 & 0 & 0 \end{bmatrix}$ and frame a, sketch frame b.



Frame 1 has been rotated away from frame 0 by 30° about z_0. Find $\vec{r}^{\,0}$ given $\vec{r}^{\,1} = [0,\ 2,\ 0]^T$, $\cos(30^\circ) = \frac{\sqrt{3}}{2}$ and $\sin(30^\circ) = \frac{1}{2}$.

Summary

Rotation matrix can be thought of in three distinct ways:

1. It describes the orientation of one coordinate frame *wrt* another coordinate frame
2. It represents a coordinate transformation that relates the coordinates of a point (e.g., P) or vector in two different frames of reference
3. It is an operator that takes a vector $\vec{p}$ and rotates it into a new vector $C\vec{p}$, both in the same coordinate frame

The End