Lecture

Navigation Mathematics: Rotation Matrices, Part II

EE 565: Position, Navigation and Timing

Lecture Notes Update on Spring 2023

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.1

Lecture Topics

Contents

.2

1 Review

Review

Rotation matrix, C_2^1

- describes orientation of frame 2 with respect to frame 1
- is of size 3×3
- is constructed via $\left[x_2^1,\ y_2^1,\ z_2^1\right] = \begin{bmatrix} x_2 \cdot x_1, & y_2 \cdot x_1, & z_2 \cdot x_1 \\ x_2 \cdot y_1, & y_2 \cdot y_1, & z_2 \cdot y_1 \\ x_2 \cdot z_1, & y_2 \cdot z_1, & z_2 \cdot z_1 \end{bmatrix}$
- has inverse $[C_2^1]^{-1} = [C_2^1]^T = C_1^2$
- is of the form $\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{z,\theta}$ for the basic (elementary) rotation about the $z-$axis by angle θ; similarly,

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix},\quad R_{y,\theta} = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

- recoordinatizes vector $\vec{v}^{\,2}$ in frame 1 via $\vec{v}^{\,1} = C_2^1 \vec{v}^{\,2}$ .3

2 Parameterizations of Rotations

Parameterizations of Rotations

Many approaches to parameterize orientation

1. Rotation matrices use $3 \times 3 = 9$ parameters
   - these 9 parameters are not independent
   - 3 constraints due to columns being orthogonal
   - 3 constraints due to columns being unit vectors

   $\Rightarrow 3$ free variables exist $\Rightarrow$ need only 3 parameters to describe orientation
2. Examples of $3-$parameter descriptions:
   - fixed-axis rotations (e.g., Roll-Pitch-Yaw/ZYX)
   - relative-axis (Euler) rotations (e.g., ZYZ, ZYX, ...)
   - angle and axis
3. Quaternions use 4 parameters .4

3 Fixed versus Relative Rotations

Fixed versus Relative Rotations

When one wants to rotate a coordinate frame about an axis, that axis can be in a fixed-frame or relative-frame.

1. Fixed-axis rotation - rotation performed about $x-$, $y-$, or $z-$axis of initial (and fixed) coordinate frame
2. Relative-axis rotation - rotation performed about $x-$, $y-$, or $z-$axis of current (and relative) coordinate frame
   - sometimes referred to as Euler rotations

Resulting orientation is quite different! .5

Notation for Rotation Matrices

C versus R

- C_b^a is a rotation matrix used to describe orientation/attitude of coordinate frame b relative to coordinate frame a
- R is a rotation matrix used to describe a specific rotation or operation, e.g., $R_{\vec{r},\beta}$ notes rotation about the unit vector $\vec{r}$ by angle β .6

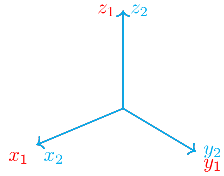
Example Sequence of Rotations

Example sequence of three consecutive rotations to compare fixed versus relative.

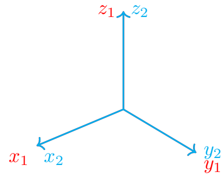
- **Step 1:** Rotate about the z-axis by ψ
- **Step 2:** Rotate about the y-axis by θ
- **Step 3:** Rotate about the x-axis by ϕ .7

Example Sequence of Rotations

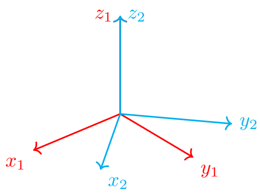
Relative Axis Rotation



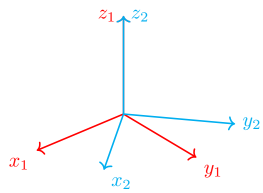
Fixed Axis Rotation



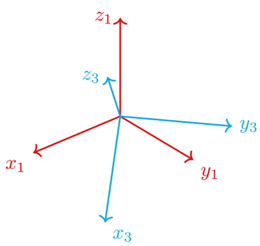
Relative Axis Rotation
Rotate about z_1



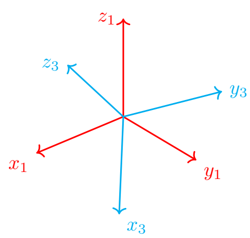
Fixed Axis Rotation
Rotate about z_1



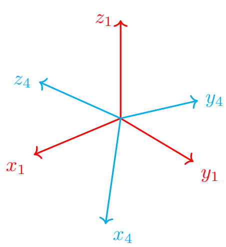
Relative Axis Rotation
Rotate about y_2



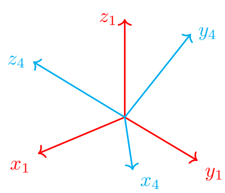
Fixed Axis Rotation
Rotate about y_1 .8



Relative Axis Rotation
Rotate about x_3



Fixed Axis Rotation
Rotate about x_1



4 Composition of Relative-axis Rotations

Composition of Relative-axis Rotations

Construct rotation matrix that represents composition of relative-axis rotations using Z-Y-X sequence of three rotations from previous example.

- Start with last rotation $C_4^3 = [x_4^3, y_4^3, z_4^3] = R_{x,\phi}$, and recall columns are vectors.
- To re-coordinatize vectors x_4^3, y_4^3, z_4^3 in frame 2, multiply each by $C_3^2 = R_{y,\theta}$.

  $\Rightarrow$ (in matrix form) $[C_3^2 x_4^3, C_3^2 y_4^3, C_3^2 z_4^3] = [x_4^2, y_4^2, z_4^2] = C_4^2$

  where it is noted that $[C_3^2 x_4^3, C_3^2 y_4^3, C_3^2 z_4^3] = C_3^2[x_4^3, y_4^3, z_4^3] = C_3^2 C_4^3 = C_4^2$

.9

Composition of Relative-axis Rotations

- To re-coordinatize vectors x_4^2, y_4^2, z_4^2 in frame 1, multiply each by $C_2^1 = R_{z,\psi}$.

  $\Rightarrow [C_2^1 x_4^2, C_2^1 y_4^2, C_2^1 z_4^2] = C_2^1 [x_4^2, y_4^2, z_4^2] = C_2^1 C_4^2 = C_2^1 C_3^2 C_4^3 = C_4^1$

- Combined sequence of relative-rotations yields

  $$C_4^1 = C_2^1 C_3^2 C_4^3 = \underbrace{R_{z,\psi}}_{1st} \underbrace{R_{y,\theta}}_{2nd} \underbrace{R_{x,\phi}}_{3rd}$$

- Note order is left to right!

- Additional relative-rotations represented by right (post) matrix multiplies.

.10

Rotation Matrix from Relative ZYX

For the relative-axis rotations Z(ψ), Y(θ), X(ϕ)

$$\begin{aligned} C_4^1 &= C_2^1 C_3^2 C_4^3 \\ &= R_{z,\psi} R_{y,\theta} R_{x,\phi} \\ &= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \\ &= \begin{bmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & c_\phi c_\psi s_\theta + s_\phi s_\psi \\ c_\theta s_\psi & c_\phi c_\psi + s_\theta s_\phi s_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix} \end{aligned}$$

where the notation $c_\beta = \cos(\beta)$ and $s_\beta = \sin(\beta)$ are introduced. .11

5 Composition of Fixed-axis Rotations

Composition of Fixed-axis Rotations

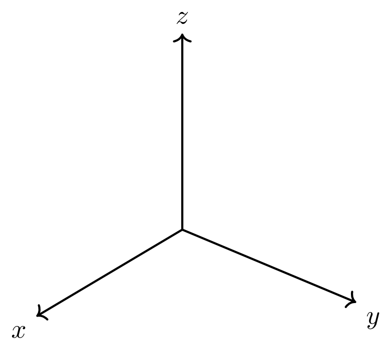
- Development of equivalent rotation matrix for sequence of fixed-axis rotations will make use of rotation matrix's ability to rotate a vector.
- A vector $\vec{p}$ can be rotated into a new vector via $R\vec{p}$, both in the same coordinate frame.
- The sequence Z(ψ) - Y(θ) - X(ϕ) aka Yaw-Pitch-Roll will be considered again, but this time about fixed-axes.

.12

Composition of Fixed-axis Rotations

Quick aside - example of rotating a vector in same coordinate system.

- Sketch $\vec{p} = [1,\ -1,\ 1]^T$ before and after its rotation about z by $90°$ (use $R_{z,90°}$ for calculation of rotated value).



.13

Composition of Fixed-axis Rotations

- First $z-$axis rotation rotates frame {1}'s basis vectors to become frame {2}'s basis vectors $[\vec{x}_2^1, \vec{y}_2^1, \vec{z}_2^1] = [R_{z,\psi}\vec{x}_1^1, R_{z,\psi}\vec{y}_1^1, R_{z,\psi}\vec{z}_1^1] = R_{z,\psi}[\vec{x}_1^1, \vec{y}_1^1, \vec{z}_1^1] = R_{z,\psi}I = R_{z,\psi}$.
- Second $y-$axis rotation rotates frame {2}'s basis vectors to become frame {3}'s basis vectors $[\vec{x}_3^1, \vec{y}_3^1, \vec{z}_3^1] = [R_{y,\theta}\vec{x}_2^1, R_{y,\theta}\vec{y}_2^1, R_{y,\theta}\vec{z}_2^1] = R_{y,\theta}[\vec{x}_2^1, \vec{y}_2^1, \vec{z}_2^1] = R_{y,\theta}R_{z,\psi}$.
- Third $x-$axis rotation rotates frame {3}'s basis vectors to become frame {4}'s basis vectors $[\vec{x}_4^1, \vec{y}_4^1, \vec{z}_4^1] = [R_{x,\phi}\vec{x}_3^1, R_{x,\phi}\vec{y}_3^1, R_{x,\phi}\vec{z}_3^1] = R_{x,\phi}[\vec{x}_3^1, \vec{y}_3^1, \vec{z}_3^1] = R_{x,\phi}R_{y,\theta}R_{z,\psi}$.
  $\Rightarrow C_4^1 = [\vec{x}_4^1, \vec{y}_4^1, \vec{z}_4^1] = \underbrace{R_{x,\phi}}_{3rd}\underbrace{R_{y,\theta}}_{2nd}\underbrace{R_{z,\psi}}_{1st}$
- Note order is right to left!
- Additional fixed-rotations represented by left (pre) matrix multiplies.

.14

Composition of Fixed-axis Rotations

For the fixed-axis rotations Z(ψ), Y(θ), X(ϕ)

$$
\begin{aligned}
C_4^1 &= R_{x,\phi}R_{y,\theta}R_{z,\psi} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\theta c_\psi & -c_\theta s_\psi & s_\theta \\ c_\psi s_\theta s_\phi + c_\phi s_\psi & c_\phi c_\psi - s_\theta s_\phi s_\psi & -c_\theta s_\phi \\ s_\phi s_\psi - c_\phi c_\psi s_\theta & c_\psi s_\phi + c_\phi s_\theta s_\psi & c_\theta c_\phi \end{bmatrix}
\end{aligned}
$$

which is quite different than the result for the same sequence of relative-axis rotations. .15

6 Example

Example

Find the rotation matrix that represents the orientation of the coordinate frame that results from the following sequence of rotations. Assume the frames start in the same orientation.

1. Rotate about fixed $x-$axis by ϕ.
2. Rotate about fixed $z-$axis by θ.
3. Rotate about current $x-$axis by ψ.
4. Rotate about current $z-$axis by α.
5. Rotate about fixed $y-$axis by β.
6. Rotate about current $y-$axis by γ.

.16

7 Summary

Fixed vs Relative Rotations

- Fixed-axis Rotations
  - Multiply on the *LEFT*
  - $C_{final} = R_n \dots R_2 R_1$

Fixed-axis Rotation
$C_{resultant} = R_{fixed} C_{original}$

- Relative-axis (Euler) Rotations
  - Multiply on the *RIGHT*
  - $C_{final} = R_1 R_2 \dots R_n$

Relative-axis Rotation
$C_{resultant} = C_{original} R_{relative}$

Two types of rotations can be composed noting order of multiplication

.17

The End

.18