EE 565: Position, Navigation and Timing

Navigation Mathematics: Rotation Matrices, Part II

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Lecture Topics

Review

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2. Examples of 3−parameter descriptions:
   - fixed-axis rotations (e.g., Roll-Pitch-Yaw/ZYX)
   - relative-axis (Euler) rotations (e.g., ZYZ, ZYX, ...)
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3. Quaternions use 4 parameters

Fixed vs Relative

When one wants to rotate a coordinate frame about an axis, that axis can be in a fixed-frame or relative-frame.

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Resulting orientation is quite different!

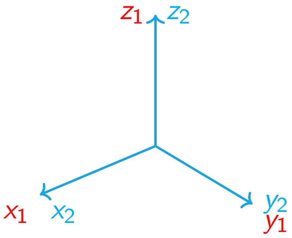
C versus R

- C_b^a is a rotation matrix used to describe orientation/attitude of coordinate frame b relative to coordinate frame a
- R is a rotation matrix used to describe a specific rotation or operation, e.g., $R_{\vec{r},\beta}$ notes rotation about the unit vector $\vec{r}$ by angle β

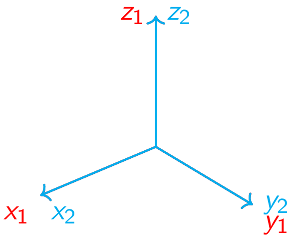
Example sequence of three consecutive rotations to compare fixed versus relative.

- **Step 1:** Rotate about the z-axis by ψ
- **Step 2:** Rotate about the y-axis by θ
- **Step 3:** Rotate about the x-axis by ϕ

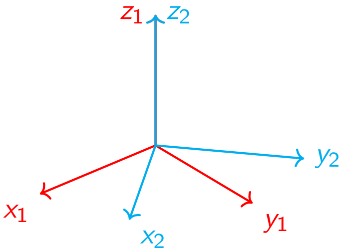
Relative-axis Rotation



Fixed-axis Rotation



Relative-axis Rotation
Rotate about z_1



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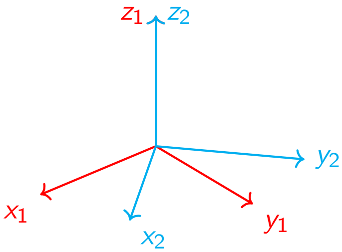
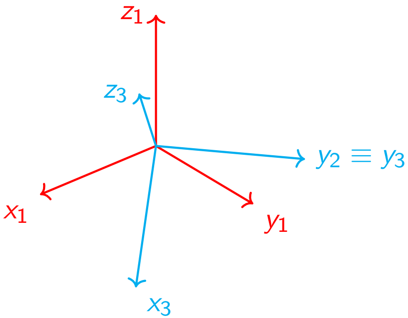


Relative-axis Rotation
Rotate about y_2



Fixed-axis Rotation
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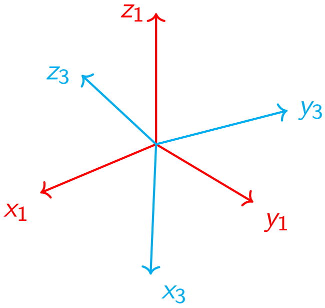
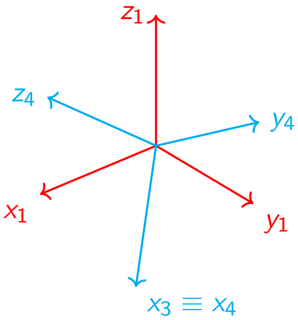


Relative-axis Rotation

Rotate about x_3



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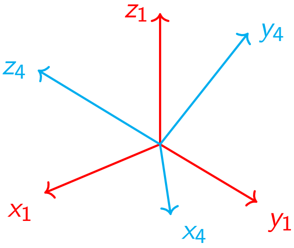

Relative-axis Rotations

Construct rotation matrix that represents composition of relative-axis rotations using Z-Y-X sequence of three rotations from previous example.

- Start with last rotation $C_4^3 = [x_4^3, y_4^3, z_4^3] = R_{x,\phi}$, and recall columns are vectors.
- To re-coordinatize vectors x_4^3, y_4^3, z_4^3 in frame 2, multiply each by $C_3^2 = R_{y,\theta}$.

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$\Rightarrow$ (in matrix form) $[C_3^2 x_4^3, C_3^2 y_4^3, C_3^2 z_4^3] = [x_4^2, y_4^2, z_4^2] = C_4^2$

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where it is noted that $[C_3^2 x_4^3, C_3^2 y_4^3, C_3^2 z_4^3] = C_3^2 [x_4^3, y_4^3, z_4^3] = C_3^2 C_4^3 = C_4^2$

- To re-coordinatize vectors x_4^2, y_4^2, z_4^2 in frame 1, multiply each by $C_2^1 = R_{z,\psi}$.

$$\Rightarrow [C_2^1 x_4^2, C_2^1 y_4^2, C_2^1 z_4^2] = C_2^1 [x_4^2, y_4^2, z_4^2] = C_2^1 C_4^2 = C_2^1 C_3^2 C_4^3 = C_4^1$$

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- Combined sequence of relative-rotations yields

$$C_4^1 = C_2^1 C_3^2 C_4^3 = \underbrace{R_{z,\psi}}_{1st} \underbrace{R_{y,\theta}}_{2nd} \underbrace{R_{x,\phi}}_{3rd}$$

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- Note order is left to right!

- Additional relative-rotations represented by right (post) matrix multiplies.

For the relative-axis rotations Z(ψ), Y(θ), X(ϕ)

$$
\begin{aligned}
C_4^1 &= C_2^1 C_3^2 C_4^3 \\
&= R_{z,\psi} R_{y,\theta} R_{x,\phi} \\
&= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \\
&= \begin{bmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & c_\phi c_\psi s_\theta + s_\phi s_\psi \\ c_\theta s_\psi & c_\phi c_\psi + s_\theta s_\phi s_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}
\end{aligned}
$$

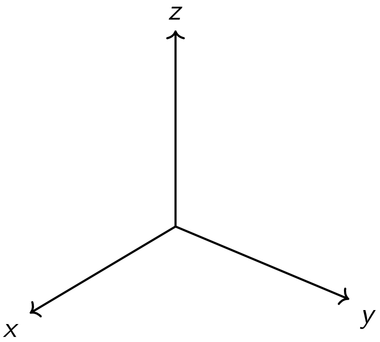
where the notation $c_\beta = \cos(\beta)$ and $s_\beta = \sin(\beta)$ are introduced.

Fixed-axis Rotations

- Development of equivalent rotation matrix for sequence of fixed-axis rotations will make use of rotation matrix's ability to rotate a vector.
- A vector $\vec{p}$ can be rotated into a new vector via $R\vec{p}$, both in the same coordinate frame.
- The sequence Z(ψ) - Y(θ) - X(ϕ) aka Yaw-Pitch-Roll will be considered again, but this time about fixed-axes.

Quick aside - example of rotating a vector in same coordinate system.

- Sketch $\vec{p} = [1,\ -1,\ 1]^T$ before and after its rotation about z by 90° (use $R_{z,90^\circ}$ for calculation of rotated value).



- First $z-$axis rotation rotates frame $\{1\}$'s basis vectors to become frame $\{2\}$'s basis vectors $[\vec{x}\,^1_2, \vec{y}\,^1_2, \vec{z}\,^1_2] = [R_{z,\psi}\vec{x}\,^1_1, R_{z,\psi}\vec{y}\,^1_1, R_{z,\psi}\vec{z}\,^1_1] = R_{z,\psi}[\vec{x}\,^1_1, \vec{y}\,^1_1, \vec{z}\,^1_1] = R_{z,\psi}I = R_{z,\psi}$.

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- Second $y-$axis rotation rotates frame $\{2\}$'s basis vectors to become frame $\{3\}$'s basis vectors $[\vec{x}^1_3, \vec{y}^1_3, \vec{z}^1_3] = [R_{y,\theta}\vec{x}^1_2, R_{y,\theta}\vec{y}^1_2, R_{y,\theta}\vec{z}^1_2] = R_{y,\theta}[\vec{x}^1_2, \vec{y}^1_2, \vec{z}^1_2] = R_{y,\theta} R_{z,\psi}$.

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- Third $x-$axis rotation rotates frame $\{3\}$'s basis vectors to become frame $\{4\}$'s basis vectors $[\vec{x}^{\,1}_{4}, \vec{y}^{\,1}_{4}, \vec{z}^{\,1}_{4}] = [R_{x,\phi}\vec{x}^{\,1}_{3}, R_{x,\phi}\vec{y}^{\,1}_{3}, R_{x,\phi}\vec{z}^{\,1}_{3}] = R_{x,\phi}[\vec{x}^{\,1}_{3}, \vec{y}^{\,1}_{3}, \vec{z}^{\,1}_{3}] = R_{x,\phi}R_{y,\theta}R_{z,\psi}$.

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$$\Rightarrow C^1_4 = [\vec{x}^1_4, \vec{y}^1_4, \vec{z}^1_4] = \underbrace{R_{x,\phi}}_{3rd}\underbrace{R_{y,\theta}}_{2nd}\underbrace{R_{z,\psi}}_{1st}$$

- First $z-$axis rotation rotates frame $\{1\}$'s basis vectors to become frame $\{2\}$'s basis vectors $[\vec{x}^{\,1}_{\,2}, \vec{y}^{\,1}_{\,2}, \vec{z}^{\,1}_{\,2}] = [R_{z,\psi}\vec{x}^{\,1}_{\,1}, R_{z,\psi}\vec{y}^{\,1}_{\,1}, R_{z,\psi}\vec{z}^{\,1}_{\,1}] = R_{z,\psi}[\vec{x}^{\,1}_{\,1}, \vec{y}^{\,1}_{\,1}, \vec{z}^{\,1}_{\,1}] = R_{z,\psi} I = R_{z,\psi}$.
- Second $y-$axis rotation rotates frame $\{2\}$'s basis vectors to become frame $\{3\}$'s basis vectors $[\vec{x}^{\,1}_{\,3}, \vec{y}^{\,1}_{\,3}, \vec{z}^{\,1}_{\,3}] = [R_{y,\theta}\vec{x}^{\,1}_{\,2}, R_{y,\theta}\vec{y}^{\,1}_{\,2}, R_{y,\theta}\vec{z}^{\,1}_{\,2}] = R_{y,\theta}[\vec{x}^{\,1}_{\,2}, \vec{y}^{\,1}_{\,2}, \vec{z}^{\,1}_{\,2}] = R_{y,\theta} R_{z,\psi}$.
- Third $x-$axis rotation rotates frame $\{3\}$'s basis vectors to become frame $\{4\}$'s basis vectors $[\vec{x}^{\,1}_{\,4}, \vec{y}^{\,1}_{\,4}, \vec{z}^{\,1}_{\,4}] = [R_{x,\phi}\vec{x}^{\,1}_{\,3}, R_{x,\phi}\vec{y}^{\,1}_{\,3}, R_{x,\phi}\vec{z}^{\,1}_{\,3}] = R_{x,\phi}[\vec{x}^{\,1}_{\,3}, \vec{y}^{\,1}_{\,3}, \vec{z}^{\,1}_{\,3}] = R_{x,\phi} R_{y,\theta} R_{z,\psi}$.

$$\Rightarrow C^1_4 = [\vec{x}^{\,1}_{\,4}, \vec{y}^{\,1}_{\,4}, \vec{z}^{\,1}_{\,4}] = \underbrace{R_{x,\phi}}_{3rd}\,\underbrace{R_{y,\theta}}_{2nd}\,\underbrace{R_{z,\psi}}_{1st}$$

- Note order is right to left!
- Additional fixed-rotations represented by left (pre) matrix multiplies.

For the fixed-axis rotations Z(ψ), Y(θ), X(ϕ)

$$
\begin{aligned}
C_4^1 &= R_{x,\phi} R_{y,\theta} R_{z,\psi} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\theta c_\psi & -c_\theta s_\psi & s_\theta \\ c_\psi s_\theta s_\phi + c_\phi s_\psi & c_\phi c_\psi - s_\theta s_\phi s_\psi & -c_\theta s_\phi \\ s_\phi s_\psi - c_\phi c_\psi s_\theta & c_\psi s_\phi + c_\phi s_\theta s_\psi & c_\theta c_\phi \end{bmatrix}
\end{aligned}
$$

For the fixed-axis rotations Z(ψ), Y(θ), X(ϕ)

$$
\begin{aligned}
C_4^1 &= R_{x,\phi} R_{y,\theta} R_{z,\psi} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}
\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}
\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\theta c_\psi & -c_\theta s_\psi & s_\theta \\ c_\psi s_\theta s_\phi + c_\phi s_\psi & c_\phi c_\psi - s_\theta s_\phi s_\psi & -c_\theta s_\phi \\ s_\phi s_\psi - c_\phi c_\psi s_\theta & c_\psi s_\phi + c_\phi s_\theta s_\psi & c_\theta c_\phi \end{bmatrix}
\end{aligned}
$$

which is quite different than the result for the same sequence of relative-axis rotations.

Example

Find the rotation matrix that represents the orientation of the coordinate frame that results from the following sequence of rotations. Assume the frames start in the same orientation.

Find the rotation matrix that represents the orientation of the coordinate frame that results from the following sequence of rotations. Assume the frames start in the same orientation.

1. Rotate about fixed $x-$axis by ϕ.

Example

Find the rotation matrix that represents the orientation of the coordinate frame that results from the following sequence of rotations. Assume the frames start in the same orientation.

1. Rotate about fixed $x-$axis by ϕ.
2. Rotate about fixed $z-$axis by θ.

Find the rotation matrix that represents the orientation of the coordinate frame that results from the following sequence of rotations. Assume the frames start in the same orientation.

1. Rotate about fixed $x-$axis by ϕ.
2. Rotate about fixed $z-$axis by θ.
3. Rotate about current $x-$axis by ψ.

Find the rotation matrix that represents the orientation of the coordinate frame that results from the following sequence of rotations. Assume the frames start in the same orientation.

1. Rotate about fixed $x-$axis by ϕ.
2. Rotate about fixed $z-$axis by θ.
3. Rotate about current $x-$axis by ψ.
4. Rotate about current $z-$axis by α.

Find the rotation matrix that represents the orientation of the coordinate frame that results from the following sequence of rotations. Assume the frames start in the same orientation.

1. Rotate about fixed $x-$axis by ϕ.
2. Rotate about fixed $z-$axis by θ.
3. Rotate about current $x-$axis by ψ.
4. Rotate about current $z-$axis by α.
5. Rotate about fixed $y-$axis by β.

Find the rotation matrix that represents the orientation of the coordinate frame that results from the following sequence of rotations. Assume the frames start in the same orientation.

1. Rotate about fixed $x-$axis by ϕ.
2. Rotate about fixed $z-$axis by θ.
3. Rotate about current $x-$axis by ψ.
4. Rotate about current $z-$axis by α.
5. Rotate about fixed $y-$axis by β.
6. Rotate about current $y-$axis by γ.

Summary

- Fixed-axis Rotations
  - Multiply on the LEFT
  - $C_{final} = R_n \dots R_2 R_1$

Fixed-axis Rotation

$$C_{resultant} = R_{fixed} C_{original}$$

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  - $C_{final} = R_n \ldots R_2 R_1$

Fixed-axis Rotation

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- Relative-axis (Euler) Rotations
  - Multiply on the RIGHT
  - $C_{final} = R_1 R_2 \ldots R_n$

Relative-axis Rotation

$$C_{resultant} = C_{original} R_{relative}$$

- Fixed-axis Rotations
  - Multiply on the LEFT
  - $C_{final} = R_n \ldots R_2 R_1$

Fixed-axis Rotation

$C_{resultant} = R_{fixed} C_{original}$

- Relative-axis (Euler) Rotations
  - Multiply on the RIGHT
  - $C_{final} = R_1 R_2 \ldots R_n$

Relative-axis Rotation

$C_{resultant} = C_{original} R_{relative}$

Two types of rotations can be composed noting order of multiplication

The End