EE 565: Position, Navigation and Timing Navigation Mathematics: Rotation Matrices, Part II

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- Parameterizations of Rotations
- Fixed versus Relative Rotations
- Opposition of Relative-axis Rotations
- 5 Composition of Fixed-axis Rotations
- 6 Example



Review	Orientation	Fixed vs Relative	Relative-axis Rotations	Fixed-axis Rotations	Example	Summary
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Review	Orientation	Fixed vs Relative	e Relative-axis Rotations	Fixed-axis Rotations	Example	Summary
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Rotation matrix, C_2^1

• describes orientation of

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Rotation matrix, C_2^1

• describes orientation of frame 2 with respect to frame 1

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- describes orientation of frame 2 with respect to frame 1
- is of size

Review	Orientation	Fixed vs Relative	Relative-axis Rotations	Fixed-axis Rotations	Example	Summary
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- describes orientation of frame 2 with respect to frame 1
- \bullet is of size 3×3

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- describes orientation of frame 2 with respect to frame 1
- \bullet is of size 3×3
- is constructed via

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• is constructed via
$$\begin{bmatrix} x_2^1, y_2^1, z_2^1 \end{bmatrix} = \begin{bmatrix} x_2 \cdot x_1, y_2 \cdot x_1, z_2 \cdot x_1 \\ x_2 \cdot y_1, y_2 \cdot y_1, z_2 \cdot y_1 \\ x_2 \cdot z_1, y_2 \cdot z_1, z_2 \cdot z_1 \end{bmatrix}$$

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• has inverse $[C_2^1]^{-1} =$

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• has inverse $[C_2^1]^{-1} = [C_2^1]^T =$

Revie	ew Orientation	Fixed vs Relati	ve Relative-axis Rotations	Fixed-axis Rotations	Example	Summary
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• has inverse $[C_2^1]^{-1} = [C_2^1]^T = C_1^2$
• is of the form $\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Review 0●	Orientation	Fixed vs Relative 00000	Relative-axis Rotations	Fixed-axis Rotations	Example 00	Summary 000
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• has inverse $\begin{bmatrix} C_2^1 \end{bmatrix}^{-1} = \begin{bmatrix} C_2^1 \end{bmatrix}^T = C_1^2$
• is of the form $\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{z,\theta}$ for the basic (elementary) rotation about the *z*-axis by angle θ





Rotation matrix, C_2^1

- describes orientation of frame 2 with respect to frame 1
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- recoordinatizes vector \vec{v}^2 in frame 1 via $\vec{v}^1 =$

Review	Orientation	Fixed vs Relative	Relative-axis Rotations	Fixed-axis Rotations	Example	Summary
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Rotation matrix, C_2^1

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• recoordinatizes vector \vec{v}^2 in frame 1 via $\vec{v}^1 = C_2^1 \vec{v}^2$

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Orientation

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() Rotation matrices use $3 \times 3 = 9$ parameters

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- **(**) Rotation matrices use $3 \times 3 = 9$ parameters
 - these 9 parameters are not independent

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- **(**) Rotation matrices use $3 \times 3 = 9$ parameters
 - these 9 parameters are not independent
 - 3 constraints due to columns being orthogonal
 - 3 constraints due to columns being unit vectors

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- **(**) Rotation matrices use $3 \times 3 = 9$ parameters
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 - \Rightarrow 3 free variables exist \Rightarrow need only 3 parameters to describe orientation



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- ② Examples of 3-parameter descriptions:
 - fixed-axis rotations (e.g., Roll-Pitch-Yaw/ZYX)
 - relative-axis (Euler) rotations (e.g., ZYZ, ZYX, ...)
 - angle and axis

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 - angle and axis
- Quaternions use 4 parameters

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Fixed vs Relative

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When one wants to rotate a coordinate frame about an axis, that axis can be in a fixed-frame or relative-frame.

• Fixed-axis rotation – rotation performed about x-, y-, or z-axis of initial (and fixed) coordinate frame

Review	Orientation	Fixed vs Relative	Relative-axis Rotations	Fixed-axis Rotations	Example	Summary
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When one wants to rotate a coordinate frame about an axis, that axis can be in a fixed-frame or relative-frame.

- Fixed-axis rotation rotation performed about x-, y-, or z-axis of initial (and fixed) coordinate frame
- **2** Relative-axis rotation rotation performed about x-, y-, or z-axis of current (and relative) coordinate frame
 - sometimes referred to as Euler rotations

		Fixed vs Relativ 0●000	e Relative-axis Rotations			
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- **2** Relative-axis rotation rotation performed about x-, y-, or z-axis of current (and relative) coordinate frame
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Resulting orientation is quite different!

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C versus R

- C_b^a is a rotation matrix used to describe orientation/attitude of coordinate frame b relative to coordinate frame a
- *R* is a rotation matrix used to describe a specific rotation or operation, e.g., $R_{\vec{r},\beta}$ notes rotation about the unit vector \vec{r} by angle β





Example sequence of three consecutive rotations to compare fixed versus relative.

- Step 1: Rotate about the *z*-axis by ψ
- **Step 2**: Rotate about the *y*-axis by θ
- **Step 3:** Rotate about the *x*-axis by ϕ

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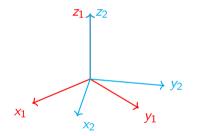
Relative-axis Rotation

Fixed-axis Rotation

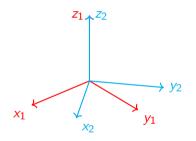


Review	Orientation	Fixed vs Relative	Relative-axis Rotations	Fixed-axis Rotations	Example	Summary
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Relative-axis Rotation Rotate about z_1



Fixed-axis Rotation Rotate about *z*₁



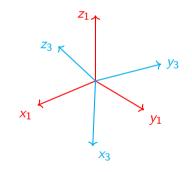
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Relative-axis Rotation Rotate about *y*₂

 $x_1 \xrightarrow{z_1} y_2 \equiv y_3$

Fixed-axis Rotation Rotate about *y*₁

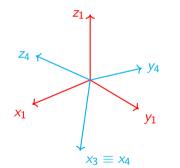


		Fixed vs Relative				
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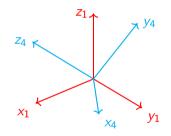




Relative-axis Rotation Rotate about x_3



Fixed-axis Rotation Rotate about x₁



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Relative-axis Rotations

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Construct rotation matrix that represents composition of relative-axis rotations using Z-Y-X sequence of three rotations from previous example.

- Start with last rotation $C_4^3 = [x_4^3, y_4^3, z_4^3] = R_{x,\phi}$, and recall columns are vectors.
- To re-coordinatize vectors x_4^3, y_4^3, z_4^3 in frame 2, multiply each by $C_3^2 = R_{y,\theta}$.

			Relative-axis Rotations			
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- To re-coordinatize vectors x_4^3, y_4^3, z_4^3 in frame 2, multiply each by $C_3^2 = R_{y,\theta}$.
 - \Rightarrow (in matrix form) $[C_3^2 x_4^3, C_3^2 y_4^3, C_3^2 z_4^3] = [x_4^2, y_4^2, z_4^2] = C_4^2$

			Relative-axis Rotations		
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- To re-coordinatize vectors x_4^3, y_4^3, z_4^3 in frame 2, multiply each by $C_3^2 = R_{y,\theta}$.
 - \Rightarrow (in matrix form) $[C_3^2 x_4^3, C_3^2 y_4^3, C_3^2 z_4^3] = [x_4^2, y_4^2, z_4^2] = C_4^2$

where it is noted that $[C_3^2 x_4^3, C_3^2 y_4^3, C_3^2 z_4^3] = C_3^2 [x_4^3, y_4^3, z_4^3] = C_3^2 C_4^3 = C_4^2$

	Relative-axis Rotations		
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Composition of Relative-axis Rotations



- To re-coordinatize vectors x_4^2, y_4^2, z_4^2 in frame 1, multiply each by $C_2^1 = R_{z,\psi}$.
 - $\Rightarrow [C_2^1 x_4^2, C_2^1 y_4^2, C_2^1 z_4^2] = C_2^1 [x_4^2, y_4^2, z_4^2] = C_2^1 C_4^2 = C_2^1 C_3^2 C_4^3 = C_4^1$

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Composition of Relative-axis Rotations



• To re-coordinatize vectors x_4^2, y_4^2, z_4^2 in frame 1, multiply each by $C_2^1 = R_{z,\psi}$.

 $\Rightarrow [C_2^1 x_4^2, C_2^1 y_4^2, C_2^1 z_4^2] = C_2^1 [x_4^2, y_4^2, z_4^2] = C_2^1 C_4^2 = C_2^1 C_3^2 C_4^3 = C_4^1$

• Combined sequence of relative-rotations yields

$$C_4^1 = C_2^1 C_3^2 C_4^3 = \underbrace{R_{z,\psi}}_{1st} \underbrace{R_{y,\theta}}_{2nd} \underbrace{R_{x,\phi}}_{3rd}$$

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Composition of Relative-axis Rotations



• To re-coordinatize vectors x_4^2, y_4^2, z_4^2 in frame 1, multiply each by $C_2^1 = R_{z,\psi}$.

 $\Rightarrow [C_2^1 x_4^2, C_2^1 y_4^2, C_2^1 z_4^2] = C_2^1 [x_4^2, y_4^2, z_4^2] = C_2^1 C_4^2 = C_2^1 C_3^2 C_4^3 = C_4^1$

• Combined sequence of relative-rotations yields

$$C_4^1 = C_2^1 C_3^2 C_4^3 = \underbrace{R_{z,\psi}}_{1st} \underbrace{R_{y,\theta}}_{2nd} \underbrace{R_{x,\phi}}_{3rd}$$

- Note order is left to right!
- Additional relative-rotations represented by right (post) matrix multiplies.

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For the relative–axis rotations $Z(\psi)$, $Y(\theta)$, $X(\phi)$

$$\begin{split} C_4^1 &= C_2^1 C_3^2 C_4^3 \\ &= R_{z,\psi} R_{y,\theta} R_{x,\phi} \\ &= \begin{bmatrix} \cos \psi & -\sin \psi & 0\\ \sin \psi & \cos \psi & 0\\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & 0 & \sin \theta\\ 0 & 1 & 0\\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0\\ 0 & \cos \phi & -\sin \phi\\ 0 & \sin \phi & \cos \phi \end{bmatrix} \\ &= \begin{bmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & c_\phi c_\psi s_\theta + s_\phi s_\psi\\ c_\theta s_\psi & c_\phi c_\psi + s_\theta s_\phi s_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi\\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix} \end{split}$$

where the notation $c_{\beta} = \cos(\beta)$ and $s_{\beta} = \sin(\beta)$ are introduced.

			Relative-axis Rotations			
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Fixed-axis Rotations

				Fixed-axis Rotations ●0000		
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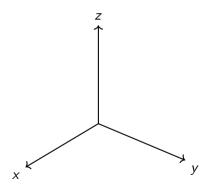
- Development of equivalent rotation matrix for sequence of fixed-axis rotations will make use of rotation matrix's ability to rotate a vector.
- A vector \vec{p} can be rotated into a new vector via $R\vec{p}$, both in the same coordinate frame.
- The sequence $Z(\psi) Y(\theta) X(\phi)$ aka Yaw-Pitch-Roll will be considered again, but this time about fixed-axes.

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Quick aside - example of rotating a vector in same coordinate system.

• Sketch $\vec{p} = [1, -1, 1]^T$ before and after its rotation about z by 90° (use $R_{z,90°}$ for calculation of rotated value).



				Fixed-axis Rotations		
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• First *z*-axis rotation rotates frame {1}'s basis vectors to become frame {2}'s basis vectors $[\vec{x}_2^1, \vec{y}_2^1, \vec{z}_2^1] = [R_{z,\psi}\vec{x}_1^1, R_{z,\psi}\vec{y}_1^1, R_{z,\psi}\vec{z}_1^1] = R_{z,\psi}[\vec{x}_1^1, \vec{y}_1^1, \vec{z}_1^1] = R_{z,\psi}I = R_{z,\psi}$.

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- First *z*-axis rotation rotates frame {1}'s basis vectors to become frame {2}'s basis vectors $[\vec{x}_2^1, \vec{y}_2^1, \vec{z}_2^1] = [R_{z,\psi}\vec{x}_1^1, R_{z,\psi}\vec{y}_1^1, R_{z,\psi}\vec{z}_1^1] = R_{z,\psi}[\vec{x}_1^1, \vec{y}_1^1, \vec{z}_1^1] = R_{z,\psi}I = R_{z,\psi}$.
- Second *y*-axis rotation rotates frame {2}'s basis vectors to become frame {3}'s basis vectors $[\vec{x}_3^1, \vec{y}_3^1, \vec{z}_3^1] = [R_{y,\theta}\vec{x}_2^1, R_{y,\theta}\vec{y}_2^1, R_{y,\theta}\vec{z}_2^1] = R_{y,\theta}[\vec{x}_2^1, \vec{y}_2^1, \vec{z}_2^1] = R_{y,\theta}R_{z,\psi}.$

				Fixed-axis Rotations 000●0		
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- First *z*-axis rotation rotates frame {1}'s basis vectors to become frame {2}'s basis vectors $[\vec{x}_2^1, \vec{y}_2^1, \vec{z}_2^1] = [R_{z,\psi}\vec{x}_1^1, R_{z,\psi}\vec{y}_1^1, R_{z,\psi}\vec{z}_1^1] = R_{z,\psi}[\vec{x}_1^1, \vec{y}_1^1, \vec{z}_1^1] = R_{z,\psi}I = R_{z,\psi}$.
- Second *y*-axis rotation rotates frame {2}'s basis vectors to become frame {3}'s basis vectors $[\vec{x}_3^1, \vec{y}_3^1, \vec{z}_3^1] = [R_{y,\theta}\vec{x}_2^1, R_{y,\theta}\vec{y}_2^1, R_{y,\theta}\vec{z}_2^1] = R_{y,\theta}[\vec{x}_2^1, \vec{y}_2^1, \vec{z}_2^1] = R_{y,\theta}R_{z,\psi}.$
- Third *x*-axis rotation rotates frame {3}'s basis vectors to become frame {4}'s basis vectors $[\vec{x}_4^1, \vec{y}_4^1, \vec{z}_4^1] = [R_{x,\phi}\vec{x}_3^1, R_{x,\phi}\vec{y}_3^1, R_{x,\phi}\vec{z}_3^1] = R_{x,\phi}[\vec{x}_3^1, \vec{y}_3^1, \vec{z}_3^1] = R_{x,\phi}R_{y,\theta}R_{z,\psi}$.

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- First *z*-axis rotation rotates frame {1}'s basis vectors to become frame {2}'s basis vectors $[\vec{x}_2^1, \vec{y}_2^1, \vec{z}_2^1] = [R_{z,\psi}\vec{x}_1^1, R_{z,\psi}\vec{y}_1^1, R_{z,\psi}\vec{z}_1^1] = R_{z,\psi}[\vec{x}_1^1, \vec{y}_1^1, \vec{z}_1^1] = R_{z,\psi}I = R_{z,\psi}$.
- Second *y*-axis rotation rotates frame {2}'s basis vectors to become frame {3}'s basis vectors $[\vec{x}_3^1, \vec{y}_3^1, \vec{z}_3^1] = [R_{y,\theta}\vec{x}_2^1, R_{y,\theta}\vec{y}_2^1, R_{y,\theta}\vec{z}_2^1] = R_{y,\theta}[\vec{x}_2^1, \vec{y}_2^1, \vec{z}_2^1] = R_{y,\theta}R_{z,\psi}.$
- Third *x*-axis rotation rotates frame {3}'s basis vectors to become frame {4}'s basis vectors $[\vec{x}_4^1, \vec{y}_4^1, \vec{z}_4^1] = [R_{x,\phi}\vec{x}_3^1, R_{x,\phi}\vec{y}_3^1, R_{x,\phi}\vec{z}_3^1] = R_{x,\phi}[\vec{x}_3^1, \vec{y}_3^1, \vec{z}_3^1] = R_{x,\phi}R_{y,\theta}R_{z,\psi}$ $\Rightarrow C_4^1 = [\vec{x}_4^1, \vec{y}_4^1, \vec{z}_4^1] = \underbrace{R_{x,\phi}}_{3rd} \underbrace{R_{y,\theta}}_{2nd} \underbrace{R_{z,\psi}}_{1st}$





- First *z*-axis rotation rotates frame {1}'s basis vectors to become frame {2}'s basis vectors $[\vec{x}_2^1, \vec{y}_2^1, \vec{z}_2^1] = [R_{z,\psi}\vec{x}_1^1, R_{z,\psi}\vec{y}_1^1, R_{z,\psi}\vec{z}_1^1] = R_{z,\psi}[\vec{x}_1^1, \vec{y}_1^1, \vec{z}_1^1] = R_{z,\psi}I = R_{z,\psi}$.
- Second *y*-axis rotation rotates frame {2}'s basis vectors to become frame {3}'s basis vectors $[\vec{x}_3^1, \vec{y}_3^1, \vec{z}_3^1] = [R_{y,\theta}\vec{x}_2^1, R_{y,\theta}\vec{y}_2^1, R_{y,\theta}\vec{z}_2^1] = R_{y,\theta}[\vec{x}_2^1, \vec{y}_2^1, \vec{z}_2^1] = R_{y,\theta}R_{z,\psi}.$
- Third *x*-axis rotation rotates frame {3}'s basis vectors to become frame {4}'s basis vectors $[\vec{x}_4^1, \vec{y}_4^1, \vec{z}_4^1] = [R_{x,\phi}\vec{x}_3^1, R_{x,\phi}\vec{y}_3^1, R_{x,\phi}\vec{z}_3^1] = R_{x,\phi}[\vec{x}_3^1, \vec{y}_3^1, \vec{z}_3^1] = R_{x,\phi}R_{y,\theta}R_{z,\psi}$ $\Rightarrow C_4^1 = [\vec{x}_4^1, \vec{y}_4^1, \vec{z}_4^1] = \underbrace{R_{x,\phi}}_{3rd} \underbrace{R_{y,\theta}}_{2nd} \underbrace{R_{z,\psi}}_{1st}$
- Note order is right to left!
- Additional fixed-rotations represented by left (pre) matrix multiplies.



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For the fixed-axis rotations $Z(\psi)$, $Y(\theta)$, $X(\phi)$

$$\begin{split} C_4^1 &= R_{\mathbf{x},\phi} R_{\mathbf{y},\theta} R_{\mathbf{z},\psi} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_\theta c_\psi & -c_\theta s_\psi & s_\theta \\ c_\psi s_\theta s_\phi + c_\phi s_\psi & c_\phi c_\psi - s_\theta s_\phi s_\psi & -c_\theta s_\phi \\ s_\phi s_\psi - c_\phi c_\psi s_\theta & c_\psi s_\phi + c_\phi s_\theta s_\psi & c_\theta c_\phi \end{bmatrix} \end{split}$$

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For the fixed-axis rotations $Z(\psi)$, $Y(\theta)$, $X(\phi)$

$$\begin{split} C_4^1 &= R_{x,\phi} R_{y,\theta} R_{z,\psi} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} c_\theta c_\psi & -c_\theta s_\psi & s_\theta \\ c_\psi s_\theta s_\phi + c_\phi s_\psi & c_\phi c_\psi - s_\theta s_\phi s_\psi & -c_\theta s_\phi \\ s_\phi s_\psi - c_\phi c_\psi s_\theta & c_\psi s_\phi + c_\phi s_\theta s_\psi & c_\theta c_\phi \end{bmatrix} \end{split}$$

which is quite different than the result for the same sequence of relative-axis rotations.



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			Example ⊙●	
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Find the rotation matrix that represents the orientation of the coordinate frame that results from the following sequence of rotations. Assume the frames start in the same orientation.

• Rotate about fixed x-axis by ϕ .

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- **•** Rotate about fixed x-axis by ϕ .
- **2** Rotate about fixed z-axis by θ .

				Example ⊙●	
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- **•** Rotate about fixed x-axis by ϕ .
- **2** Rotate about fixed z-axis by θ .
- **③** Rotate about current x-axis by ψ .

Review	Orientation	Fixed vs Relative	Relative-axis Rotations	Fixed-axis Rotations	Example	Summary
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- Rotate about fixed x-axis by ϕ .
- **2** Rotate about fixed z-axis by θ .
- **③** Rotate about current x-axis by ψ .
- **(9)** Rotate about current z-axis by α .

Review	Orientation	Fixed vs Relative	e Relative-axis Rotations	Fixed-axis Rotations	Example	Summary
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- Rotate about fixed x-axis by ϕ .
- **2** Rotate about fixed z-axis by θ .
- **③** Rotate about current x-axis by ψ .
- **(**) Rotate about current z-axis by α .
- **(a)** Rotate about fixed y-axis by β .

Review	Orientation	Fixed vs Relative	Relative-axis Rotations	Fixed-axis Rotations	Example	Summary
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- **•** Rotate about fixed x-axis by ϕ .
- **2** Rotate about fixed z-axis by θ .
- **③** Rotate about current x-axis by ψ .
- **(**) Rotate about current z-axis by α .
- **(**) Rotate about fixed y-axis by β .
- Rotate about current y-axis by γ .

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Fixed vs Relative Rotations

- Fixed-axis Rotations
 - Multiply on the LEFT
 - $C_{final} = R_n \dots R_2 R_1$

Fixed-axis Rotation

 $C_{resultant} = R_{fixed} C_{original}$





Fixed vs Relative Rotations

- Fixed-axis Rotations
 - Multiply on the LEFT
 - $C_{final} = R_n \dots R_2 R_1$

Fixed-axis Rotation

 $C_{resultant} = R_{fixed} C_{original}$

- Relative-axis (Euler) Rotations
 - Multiply on the **RIGHT**
 - $C_{final} = R_1 R_2 \dots R_n$

Relative-axis Rotation

 $C_{resultant} = C_{original} R_{relative}$





Fixed vs Relative Rotations

- Fixed-axis Rotations
 - Multiply on the LEFT
 - $C_{final} = R_n \dots R_2 R_1$

Fixed-axis Rotation

 $C_{resultant} = R_{fixed} C_{original}$

- Relative-axis (Euler) Rotations
 - Multiply on the **RIGHT**
 - $C_{final} = R_1 R_2 \dots R_n$

Relative-axis Rotation

 $C_{resultant} = C_{original} R_{relative}$

Two types of rotations can be composed noting order of multiplication







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