EE 565: Position, Navigation and Timing

Navigation Mathematics: Other Descriptions of Orientation

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Lecture Topics

Review

Rotation Matrices R, C

- Notation to be adopted:
    - C represents an orientation
    - R represents a rotation

Rotation Matrices R, C

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$$C_{final} = C_{initial} R$$

  - rotations about fixed axis $\Rightarrow$ pre-/left-multiply

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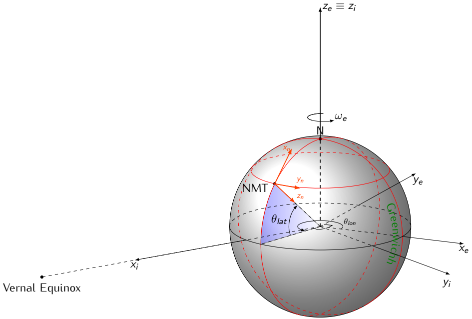
$$C_{final} = R C_{initial}$$

- $3 \times 3 = 9$ elements with 6 constraints $\Rightarrow$ 3 parameters are sufficient to describe orientation

What is orientation of ECEF frame resolved in ECI frame, i.e. C_e^i?

$$C_e^i = R_{z,\theta_{ie}} = \begin{bmatrix} \cos\theta_{ie} & -\sin\theta_{ie} & 0 \\ \sin\theta_{ie} & \cos\theta_{ie} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

What is θ_{ie}?



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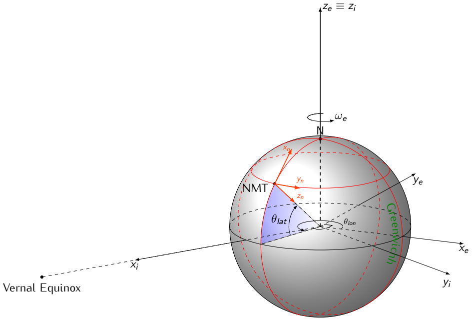

What is θ_{ie}? angle from frame $\{i\}$ to frame $\{e\}$;

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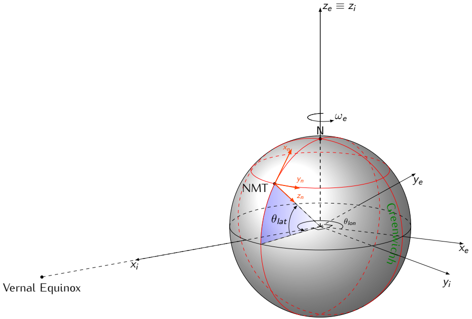
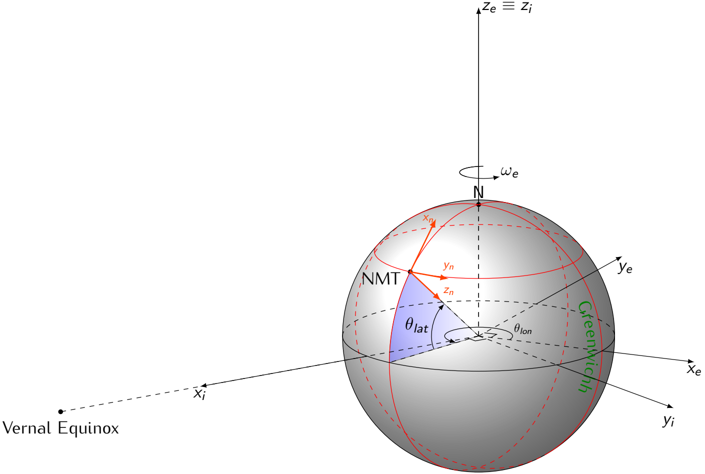


What is θ_{ie}? angle from frame $\{i\}$ to frame $\{e\}$; here $\theta_{ie} = \omega_{ie}(t - t_0)$

What is the nav frame resolved in the ECEF frame, i.e. C_n^e?



Roll-Pitch-Yaw Angles

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Roll-Pitch-Yaw angles

- often used to represent orientation of aircraft
- three angles $(\phi,\ \theta,\ \psi)$ that represent the sequence of rotations about the $x-$, $y-$ and $z-$axes of a fixed frame
- given angles $(\phi,\ \theta,\ \psi)$, equivalent rotation matrix can be found via

$$
\begin{aligned}
C_{RPY} &= R_{z,\psi} R_{y,\theta} R_{x,\phi} \\
&= \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{bmatrix} \\
&= \begin{bmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & c_\phi c_\psi s_\theta + s_\phi s_\psi \\ c_\theta s_\psi & c_\phi c_\psi + s_\theta s_\phi s_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}
\end{aligned}
$$

Given a rotation matrix that describes a desired orientation

$$C_{desired} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Roll-Pitch-Yaw angles $(\phi,\ \theta,\ \psi)$ can be found (the inverse solution) by equating combinations of terms

$$\begin{bmatrix} \boxed{c_\theta c_\psi} & c_\psi s_\theta s_\phi - c_\phi s_\psi & c_\phi c_\psi s_\theta + s_\phi s_\psi \\ \boxed{c_\theta s_\psi} & c_\phi c_\psi + s_\theta s_\phi s_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix} = \begin{bmatrix} \boxed{C_{11}} & C_{12} & C_{13} \\ \boxed{C_{21}} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$\frac{C_{21}}{C_{11}} = \frac{c_\theta s_\psi}{c_\theta c_\psi} = \tan(\psi)$$

$$\begin{bmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & c_\phi c_\psi s_\theta + s_\phi s_\psi \\ c_\theta s_\psi & c_\phi c_\psi + s_\theta s_\phi s_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi \\ -s_\theta & \boxed{c_\theta s_\phi} & \boxed{c_\theta c_\phi} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & \boxed{C_{32}} & \boxed{C_{33}} \end{bmatrix}$$

$$\frac{C_{32}}{C_{33}} = \frac{c_\theta s_\phi}{c_\theta c_\phi} = \tan(\phi)$$

$$\begin{bmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & c_\phi c_\psi s_\theta + s_\phi s_\psi \\ c_\theta s_\psi & c_\phi c_\psi + s_\theta s_\phi s_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi \\ -s_\theta & \boxed{c_\theta s_\phi} & \boxed{c_\theta c_\phi} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & \boxed{C_{32}} & \boxed{C_{33}} \end{bmatrix}$$

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$$\begin{bmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & c_\phi c_\psi s_\theta + s_\phi s_\psi \\ c_\theta s_\psi & c_\phi c_\psi + s_\theta s_\phi s_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi \\ \boxed{-s_\theta} & \boxed{c_\theta s_\phi} & \boxed{c_\theta c_\phi} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ \boxed{C_{31}} & \boxed{C_{32}} & \boxed{C_{33}} \end{bmatrix}$$

$$\frac{-C_{31}}{\sqrt{C_{32}^2 + C_{33}^2}} = \frac{-(-s_\theta)}{\sqrt{c_\theta^2(s_\phi^2 + c_\phi^2)}} = \frac{s_\theta}{c_\theta} = \tan(\theta)$$

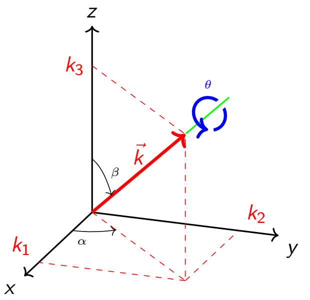
Angle-Axis

Angle-Axis

- one rotation about general axis will be used to describe orientation, so **does not** have the "rotation in sequence" issue
- rotation matrix C can be realized via rotation away from initial frame by angle θ about appropriately chosen axis $\vec{k} = [k_1,\ k_2,\ k_3]^T$ of rotation
- assume $\vec{k}$ is a unit vector

- Rotation matrix can be derived by rotating one of the principal axis (x, y, or z) onto the vector $\vec{k}$, performing a rotation of θ, and finally undoing the original changes.
- Common sequence is

$$R_{\vec{k},\theta} = \underbrace{R_{z,\alpha} \quad R_{y,\beta}}_{\text{align z with } \vec{k}} \quad R_{z,\theta} \qquad \underbrace{R_{y,-\beta} \quad R_{z,-\alpha}}_{\text{put frame back relative to } \vec{k}}$$



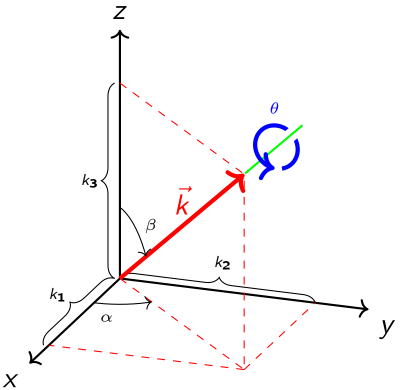
Noting

$$\sin\alpha = \frac{k_2}{\sqrt{k_1^2 + k_2^2}}, \quad \cos\alpha = \frac{k_1}{\sqrt{k_1^2 + k_2^2}}$$

$$\sin\beta = \sqrt{k_1^2 + k_2^2}, \quad \cos\beta = k_3$$

the composition of rotations becomes

$$R_{\vec{k},\theta} = \begin{bmatrix} k_1^2 V_\theta + c_\theta & k_1 k_2 V_\theta - k_3 s_\theta & k_1 k_3 V_\theta + k_2 s_\theta \\ k_1 k_2 V_\theta + k_3 s_\theta & k_2^2 V_\theta + c_\theta & k_2 k_3 V_\theta - k_1 s_\theta \\ k_1 k_3 V_\theta - k_2 s_\theta & k_2 k_3 V_\theta + k_1 s_\theta & k_3^2 V_\theta + c_\theta \end{bmatrix} \quad (1)$$



where $versin(\theta) = V_\theta \equiv 1 - c_\theta$.

Alternate approach to development of angle-axis is to relate rotation matrix to its equivalent angle-axis pair by

$$R_{\vec{k},\theta(t)} = e^{\kappa\theta(t)}$$

where

skew-symmetric

$$\kappa = [\vec{k}\times] = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix}$$

is the skew-symmetric matrix version of the axis vector $\vec{k} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}^T$ and $\kappa^T = -\kappa$.

- Using Taylor expansion of matrix-exponential

$$R_{\vec{k},\theta(t)} = e^{\kappa\theta(t)} = \mathcal{I} + \kappa\theta(t) + \frac{\kappa^2\theta^2(t)}{2!} + \frac{\kappa^3\theta^3(t)}{3!} + \cdots$$

which, after a bit of manipulation (recalling Taylor series of sine and cosine and noting $\kappa^3 = -\kappa$), can be shown to be

Rodrigues Formula

$$R_{\vec{k},\theta(t)} = \mathcal{I} + \sin(\theta(t))\kappa + [1 - \cos(\theta(t))]\,\kappa^2$$

- Multiplying out the right hand side of the above equation gives us the same rotation matrix as that in Eq. 1 shown previously.

Desired rotation matrix to $(\vec{k}, \theta)$ – the inverse problem

$$R_{\vec{k},\theta} = \begin{bmatrix} k_1^2 V_\theta + c_\theta & k_1 k_2 V_\theta - k_3 s_\theta & k_1 k_3 V_\theta + k_2 s_\theta \\ k_1 k_2 V_\theta + k_3 s_\theta & k_2^2 V_\theta + c_\theta & k_2 k_3 V_\theta - k_1 s_\theta \\ k_1 k_3 V_\theta - k_2 s_\theta & k_2 k_3 V_\theta + k_1 s_\theta & k_3^2 V_\theta + c_\theta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = R_{desired}$$

- find angle-axis pair $(\vec{k}, \theta)$ needed to realize desired rotation matrix

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- find angle-axis pair $(\vec{k}, \theta)$ needed to realize desired rotation matrix
- look at trace of rotation matrix and recall $V_\theta \equiv 1 - \cos\theta$

$$Tr\left(R_{\vec{k},\theta}\right) = \left[k_1^2 + k_2^2 + k_3^2\right](1 - \cos\theta) + 3\cos\theta = 1 + 2\cos\theta$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{Tr\left(R_{\vec{k},\theta}\right) - 1}{2}\right) = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right)$$

Now for the axis of rotation; a review of the structure suggests

$$
\begin{aligned}
r_{32} - r_{23} &= 2k_1 s_\theta \\
r_{13} - r_{31} &= 2k_2 s_\theta \\
r_{21} - r_{12} &= 2k_3 s_\theta
\end{aligned}
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$$\Rightarrow \vec{k} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \frac{1}{2s_\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

A satellite orbiting the earth can be made to point it's telescope at a desired star by performing the following motions

1. Rotate about it's x-axis by $-30°$, then
2. Rotate about it's new z-axis by $50°$, then finally
3. Rotate about it's initial y-axis by $40°$.

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What is its final orientation *wrt* the starting orientation?

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$$C_{final}^{start} = R_{(\vec{y},\mathbf{40}^\circ)} R_{(\vec{x},-\mathbf{30}^\circ)} R_{(\vec{z},\mathbf{50}^\circ)}$$

$$= \begin{bmatrix} 0.766044 & 0 & 0.642788 \\ 0 & 1 & 0 \\ -0.642788 & 0 & 0.766044 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.866025 & 0.5 \\ 0 & -0.5 & 0.866025 \end{bmatrix} \begin{bmatrix} 0.642788 & -0.766044 & 0 \\ 0.766044 & 0.642788 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.246202 & -0.793412 & 0.55667 \\ 0.663414 & 0.663414 & 0.5 \\ -0.706588 & 0.246202 & 0.246202 \end{bmatrix}$$

- In order to save energy it is desirable to perform this change in orientation with only one rotation — How?

Angle-Axis - Example

- In order to save energy it is desirable to perform this change in orientation with only one rotation — How?
- Perform a single, equivalent angle-axis rotation with

$$\theta = \cos^{-1}\left(\frac{Tr\left(C_{final}^{start}\right) - 1}{2}\right) = 76.5^\circ$$

$$\vec{k} = \frac{1}{2s_\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} = \begin{bmatrix} -0.130495 \\ 0.649529 \\ 0.749055 \end{bmatrix}$$

Angle-Axis representation can be made three parameters via

$$\vec{K} = \theta \vec{k}$$

such that

$$\theta = \|\vec{K}\|$$

and

$$\vec{k} = \frac{\vec{K}}{\|\vec{K}\|}$$

Quaternions

Euler angles, RPY angles and angle-axis consist three elements, but they are not unique, e.g., there are orientations that are represented by different Euler angles, RYP angles and angle-axis.

Quaternion

- Quaternions are 4-element representation of the rotation vectors where the additional element makes quaternions unique.
- With 4 elements quaternions have the lowest dimensionality possible for a globally nonsingular attitude representation.

Given an angle-axis pair $(\theta, \vec{k})$ or the corresponding rotation vector $\vec{K} = \theta\vec{k}$, a quaternion is defined as

$$\bar{q} = \begin{bmatrix} q_s \\ \vec{q} \end{bmatrix} = \begin{bmatrix} q_s \\ q_x \\ q_y \\ q_z \end{bmatrix} = \begin{bmatrix} \cos(\frac{\theta}{2}) \\ \vec{k}\sin(\frac{\theta}{2}) \end{bmatrix}$$

where

- $q_s = \cos(\frac{\theta}{2})$ is the scalar component
- $\vec{q} = [q_x,\ q_y,\ q_z]^T = \vec{k}\sin(\frac{\theta}{2})$ is the vector component
- $|\bar{q}| = \sqrt{q_s^2 + q_x^2 + q_y^2 + q_z^2} = \sqrt{(\cos(\frac{\theta}{2}))^2 + (k_1\sin(\frac{\theta}{2}))^2 + (k_2\sin(\frac{\theta}{2}))^2 + (k_3\sin(\frac{\theta}{2}))^2} = 1 \Rightarrow$ a unit quaternion

Trig identities can be applied term-by-term to $R_{\vec{k},\theta}$ to find $R_{\bar{q}}$.

$$
\begin{aligned}
r_{11} &= k_1^2 V_\theta + c_\theta \\
&= k_1^2(1 - \cos(\theta)) + \cos(\theta) \\
&= 2k_1^2 \underbrace{\left(\frac{1-\cos(\theta)}{2}\right)}_{\sin^2(\frac{\theta}{2})} + \underbrace{\cos(\theta)}_{\cos^2(\frac{\theta}{2}) - \sin^2(\frac{\theta}{2})} \\
&= \cos^2(\frac{\theta}{2}) + (2k_1^2 - \underbrace{1}_{k_1^2+k_2^2+k_3^2}) \sin^2(\frac{\theta}{2}) \\
&= \cos^2(\frac{\theta}{2}) + (2k_1^2 - k_1^2 - k_2^2 - k_3^2) \sin^2(\frac{\theta}{2}) \\
&= \cos^2(\frac{\theta}{2}) + (k_1^2 - k_2^2 - k_3^2) \sin^2(\frac{\theta}{2}) \\
&= \underbrace{\cos^2(\frac{\theta}{2})}_{q_s^2} + \underbrace{k_1^2 \sin^2(\frac{\theta}{2})}_{q_x^2} - \underbrace{k_2^2 \sin^2(\frac{\theta}{2})}_{q_y^2} - \underbrace{k_3^2 \sin^2(\frac{\theta}{2})}_{q_z^2} \\
&= q_s^2 + q_x^2 - q_y^2 - q_z^2
\end{aligned}
$$

Quaternion to Rotation Matrix

$$R_{\bar{q}} = \begin{bmatrix} q_s^2 + q_x^2 - q_y^2 - q_z^2 & & \\ & & \\ & & \end{bmatrix}$$

$$
\begin{aligned}
r_{12} &= k_1 k_2 V_\theta - k_3 s_\theta \\
&= k_1 k_2 (1 - \cos(\theta)) \quad - \quad k_3 \sin(\theta) \\
&= 2 k_1 k_2 \underbrace{\left(\frac{1 - \cos(\theta)}{2}\right)}_{\sin^2(\frac{\theta}{2})} \quad - \quad k_3 \quad \underbrace{\sin(\theta)}_{2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})} \\
&= 2 \underbrace{k_1 \sin(\frac{\theta}{2})}_{q_x} \underbrace{k_2 \sin(\frac{\theta}{2})}_{q_y} \quad - \quad 2 \underbrace{\cos(\frac{\theta}{2})}_{q_s} \underbrace{k_3 \sin(\frac{\theta}{2})}_{q_z} \\
&= 2(q_x q_y - q_s q_z)
\end{aligned}
$$

$$R_{\bar{q}} = \begin{bmatrix} q_s^2 + q_x^2 - q_y^2 - q_z^2 & 2(q_x q_y - q_s q_z) & \\ & & \\ & & \end{bmatrix}$$

and so on ...

Rotation matrix from given quaternion

$$R_{\bar{q}} = \begin{bmatrix} q_s^2 + q_x^2 - q_y^2 - q_z^2 & 2(q_x q_y - q_s q_z) & 2(q_x q_z + q_s q_y) \\ 2(q_x q_y + q_s q_z) & q_s^2 - q_x^2 + q_y^2 - q_z^2 & 2(q_y q_z - q_s q_x) \\ 2(q_x q_z - q_s q_y) & 2(q_y q_z + q_s q_x) & q_s^2 - q_x^2 - q_y^2 + q_z^2 \end{bmatrix}$$

Quaternion from given rotation matrix

$$R_{\bar{q}} = \begin{bmatrix} q_s^2 + q_x^2 - q_y^2 - q_z^2 & 2(q_x q_y - q_s q_z) & 2(q_x q_z + q_s q_y) \\ 2(q_x q_y + q_s q_z) & q_s^2 - q_x^2 + q_y^2 - q_z^2 & 2(q_y q_z - q_s q_x) \\ 2(q_x q_z - q_s q_y) & 2(q_y q_z + q_s q_x) & q_s^2 - q_x^2 - q_y^2 + q_z^2 \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = R_{desired}$$

$$\Rightarrow q_s = \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}} \text{ and } \vec{q} = \frac{1}{4q_s}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$$

where we have to be careful when $\theta = 180^\circ$ and $q_s = 0$. Note issue is with conversion rather than quaternions or rotation matrices.

Quaternions can be used to describe orientation and compose rotations like rotation matrices

- $C_b^a \Leftrightarrow \bar{q}\,_b^a$
- $C_f^i = R_2 R_1 R_3 \Leftrightarrow \bar{q}\,_f^i = \bar{q}_2 \otimes \bar{q}_1 \otimes \bar{q}_3$

- Quaternion inverse or conjugate

$$\bar{q}^{\,-1} = \bar{q}^{\,*} = \begin{bmatrix} q_s \\ -q_x \\ -q_y \\ -q_z \end{bmatrix}$$

- Vector transformation (change of coordinates)
  Define a "pure" vector

$$\breve{v} = \begin{bmatrix} 0 \\ \vec{v} \end{bmatrix}$$

then a vector $\vec{v}^{\,p}$ written in the p–frame may be transformed to the i–frame using

$$\breve{v}^i = \bar{q} \otimes \breve{v}^p \otimes \bar{q}^{\,-1}$$

Quaternion multiplication - first type $\otimes$

$$\bar{r} = \bar{q} \otimes \bar{p} = [\bar{q}\otimes]\bar{p} = \begin{bmatrix} q_s p_s - \vec{q} \cdot \vec{p} \\ q_s \vec{p} + p_s \vec{q} + \vec{q} \times \vec{p} \end{bmatrix}$$

where implementation via matrix multiplication achieved by defining

$$[\bar{q}\otimes] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & -q_z & q_y \\ q_y & q_z & q_s & -q_x \\ q_z & -q_y & q_x & q_s \end{bmatrix}$$

Note multiplication does not commute.

Quaternion multiplication - second type $\circledast$ (useful to re-order multiplication when certain factorizations and coordinatizations needed)

$$\bar{r} = \bar{q} \circledast \bar{p} = [\bar{q}\circledast]\bar{p} = \begin{bmatrix} q_s p_s - \vec{q} \cdot \vec{p} \\ q_s \vec{p} + p_s \vec{q} \boxed{-} \vec{q} \times \vec{p} \end{bmatrix}$$

where

$$\bar{q} \otimes \bar{p} = \bar{p} \circledast \bar{q}$$

and

$$[\bar{q}\circledast] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & q_z & -q_y \\ q_y & -q_z & q_s & q_x \\ q_z & q_y & -q_x & q_s \end{bmatrix}$$

Identities for quaternions

$$[\bar{q}^{\,-1}\otimes] = [\bar{q}\otimes]^{-1} = [\bar{q}\otimes]^T$$

$$[\bar{q}^{\,-1}\circledast] = [\bar{q}\circledast]^{-1} = [\bar{q}\circledast]^T$$

$$[\bar{q}\otimes] = e^{\frac{1}{2}[\breve{k}\otimes]} = \cos(\theta/2)\mathcal{I} + \frac{1}{2}[\breve{k}\otimes]\frac{\sin(\theta/2)}{\theta/2}$$

$$[\bar{q}\circledast] = e^{\frac{1}{2}[\breve{k}\circledast]} = \cos(\theta/2)\mathcal{I} + \frac{1}{2}[\breve{k}\circledast]\frac{\sin(\theta/2)}{\theta/2}$$

$$[\bar{q}\otimes][\bar{q}\circledast]^{-1} = [\bar{q}\circledast]^{-1}[\bar{q}\otimes] = \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{T}(\bar{q}) \end{bmatrix}$$

$$\bar{q} \otimes \bar{p} \otimes \bar{r} = (\bar{q} \otimes \bar{p}) \otimes \bar{r} = \bar{q} \otimes (\bar{p} \otimes \bar{r})$$

$$\bar{q} \circledast \bar{p} \circledast \bar{r} = (\bar{q} \circledast \bar{p}) \circledast \bar{r} = \bar{q} \circledast (\bar{p} \circledast \bar{r})$$

$$(\bar{q} \circledast \bar{p}) \otimes \bar{r} \neq \bar{q} \circledast (\bar{p} \otimes \bar{r})$$

$$(\bar{q} \otimes \bar{p}) \circledast \bar{r} \neq \bar{q} \otimes (\bar{p} \circledast \bar{r})$$

The End