EE 565: Position, Navigation and Timing

Navigation Mathematics: Translation

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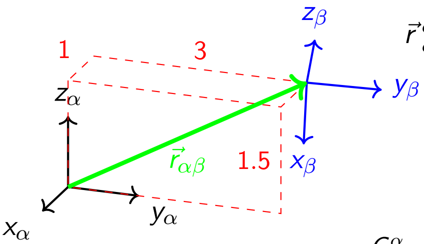
1. Vector Notation for Translation
2. Translation Between More Than Two Coordinate Frames
3. Example

Vector Notation for Translation

Define the vector $\vec{r}_{\alpha\beta}$ from the origin of $\{\alpha\}$ to the origin of $\{\beta\}$.

- specifies translation between frames



$$\vec{r}^{\,\alpha}_{\alpha\beta} = \begin{bmatrix} x^\alpha_{\alpha\beta} \\ y^\alpha_{\alpha\beta} \\ z^\alpha_{\alpha\beta} \end{bmatrix} = \begin{bmatrix} -1.00 \\ 3 \\ 1.50 \end{bmatrix}$$

$$C^\alpha_\beta = \begin{bmatrix} 0.785 & -0.366 & 0.500 \\ 0.242 & 0.924 & 0.296 \\ -0.571 & -0.111 & 0.813 \end{bmatrix}$$

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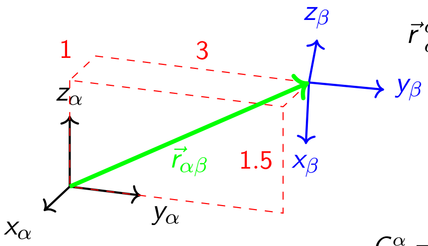

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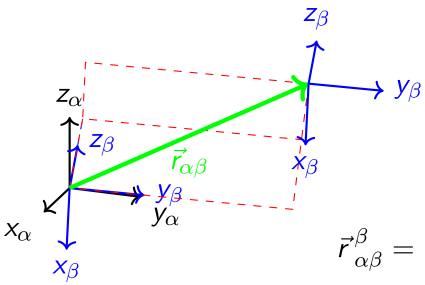
Now have means (and notation) to describe rotation and translation between coordinate frames.

- Resolve, i.e., coordinatize, $\vec{r}_{\alpha\beta}$ *wrt* frame $\{\beta\}$.



$$\vec{r}^{\beta}_{\alpha\beta} = \begin{bmatrix} x^{\beta}_{\alpha\beta} \\ y^{\beta}_{\alpha\beta} \\ z^{\beta}_{\alpha\beta} \end{bmatrix} = C^{\beta}_{\alpha} \vec{r}^{\alpha}_{\alpha\beta} = \begin{bmatrix} -0.914 \\ 2.97 \\ 1.61 \end{bmatrix}$$

Same vector, so same “direction” and length.

Reverse vector $\vec{r}$, i.e., now from origin of $\{\beta\}$ to origin of $\{\alpha\}$.

- notation:

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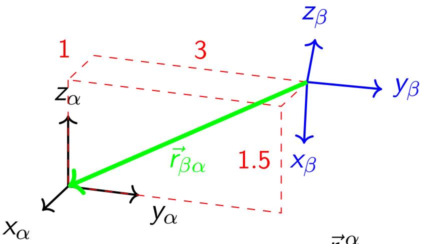
Reverse vector $\vec{r}$, i.e., now from origin of $\{\beta\}$ to origin of $\{\alpha\}$.

- notation: $\vec{r}_{\beta\alpha} = -\vec{r}_{\alpha\beta}$

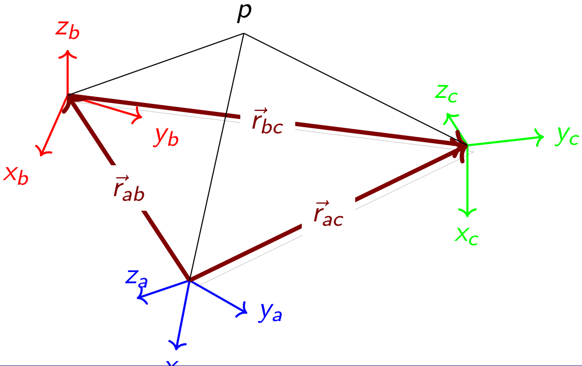


$$\vec{r}^{\,\alpha}_{\beta\alpha} = \begin{bmatrix} x^\alpha_{\beta\alpha} \\ y^\alpha_{\beta\alpha} \\ z^\alpha_{\beta\alpha} \end{bmatrix} = -\vec{r}^{\,\alpha}_{\alpha\beta} = \begin{bmatrix} -(-1.00) \\ -(3) \\ -(1.50) \end{bmatrix}$$

Translation Between More Than Two Coordinate Frames

Consider three coordinate systems $\{a\}$, $\{b\}$, $\{c\}$ that have translation and rotation relative to each other.

- Knowing relationships between frames $\{a\}$, $\{b\}$, and $\{c\}$, i.e., $\vec{r}_{ab}$, $\vec{r}_{bc}$, $\vec{r}_{ac}$, C_b^a, C_c^b, and C_c^a, location of point p can be described in any frame, i.e., $\vec{p}^{\,a}$ or $\vec{p}^{\,b}$ or $\vec{p}^{\,c}$.



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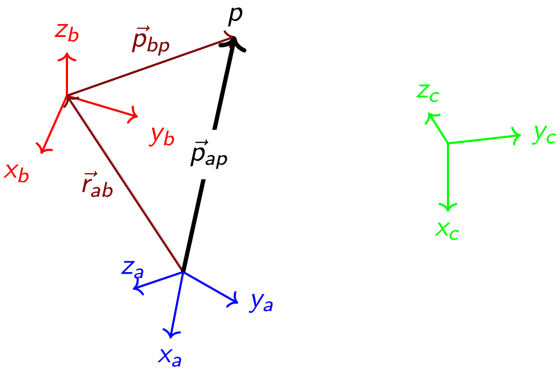
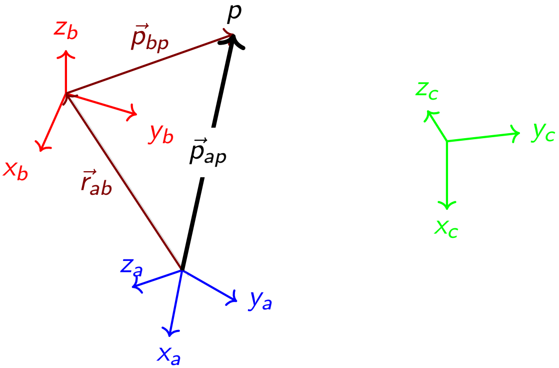
Determine the location of the point p relative to $\{a\}$ given location of point p is known relative to $\{b\}$.



- $\vec{p}_{ap} =$

Determine the location of the point p relative to $\{a\}$ given location of point p is known relative to $\{b\}$.



- $\vec{p}_{ap} = \vec{r}_{ab} + \vec{p}_{bp}$

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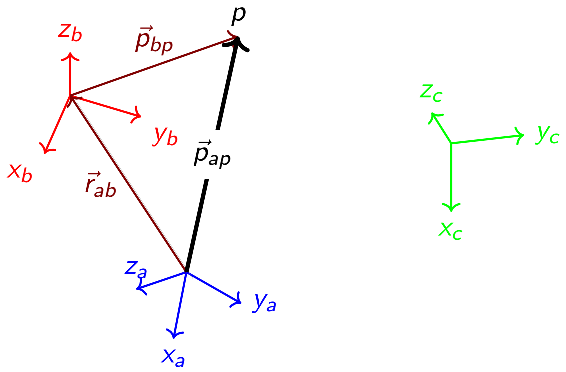

- $\vec{p}_{ap} = \vec{r}_{ab} + \vec{p}_{bp}$ In what frame?

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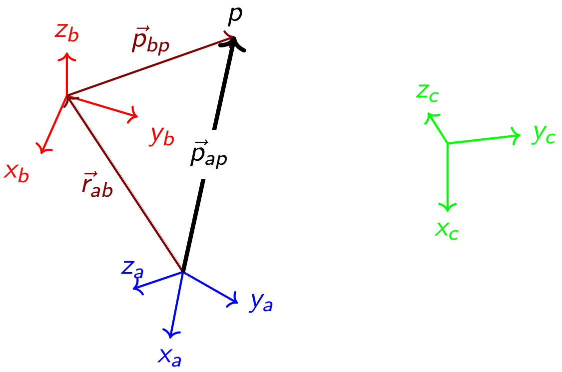

- $\vec{p}_{ap} = \vec{r}_{ab} + \vec{p}_{bp}$ In what frame?
- $\vec{p}^{\,a}_{ap} = \vec{r}^{\,a}_{ab} + \vec{p}^{\,a}_{bp}$ or
  $\vec{p}^{\,b}_{ap} = \vec{r}^{\,b}_{ab} + \vec{p}^{\,b}_{bp}$ or
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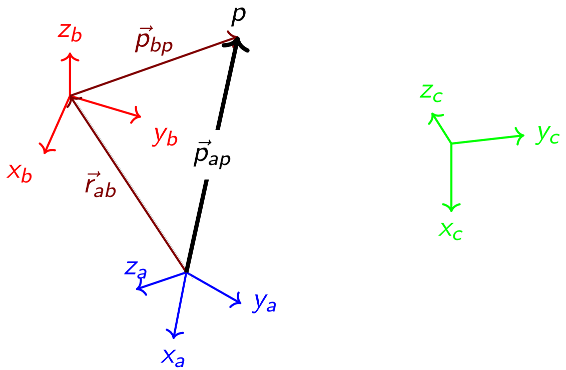

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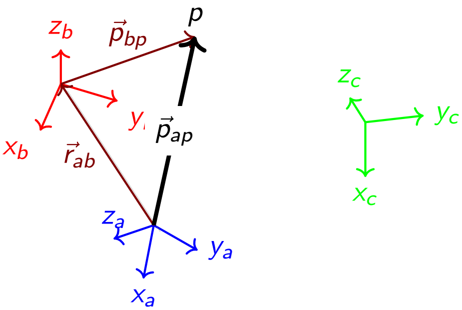
Shorthand notation: $\vec{p}^{\,a} \equiv \vec{p}^{\,a}_{ap}$

Given $\vec{p}^{\,a}_{ap} = \vec{r}^{\,a}_{ab} + \vec{p}^{\,a}_{bp}$ and/or the diagram, how would one find $\vec{p}^{\,b}_{bp}$?



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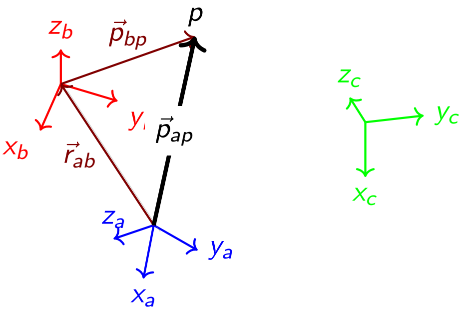

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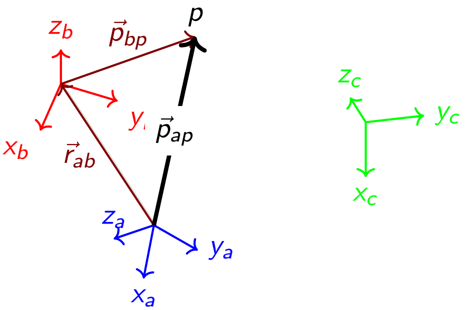

- use given relationship or vector addition

$$\Rightarrow \vec{p}^{\,a}_{bp} = \vec{p}^{\,a}_{ap} - \vec{r}^{\,a}_{ab}$$

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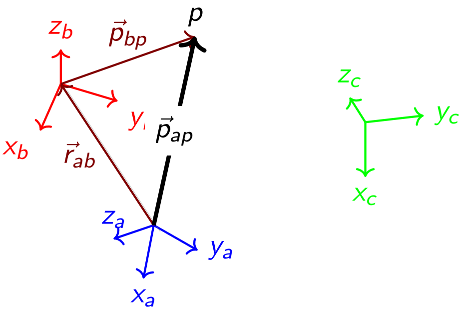

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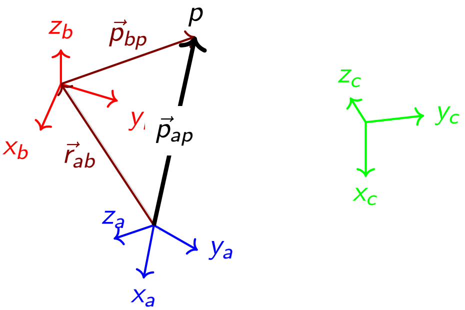

- use given relationship or vector addition

  $\Rightarrow \vec{p}^{\,a}_{bp} = \vec{p}^{\,a}_{ap} - \vec{r}^{\,a}_{ab}$
- now need to reference to $\{b\}$

  $C^b_a \vec{p}^{\,a}_{bp} = C^b_a \left(\vec{p}^{\,a}_{ap} - \vec{r}^{\,a}_{ab}\right)$

  $\Rightarrow \vec{p}^{\,b}_{bp} = \vec{p}^{\,b}_{ap} - \vec{r}^{\,b}_{ab}$

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- Recoordinatizing: $\vec{p}^{\,c}_{ap} = C^c_a \vec{p}^{\,a}_{ap}$
  (only frame of reference changes)

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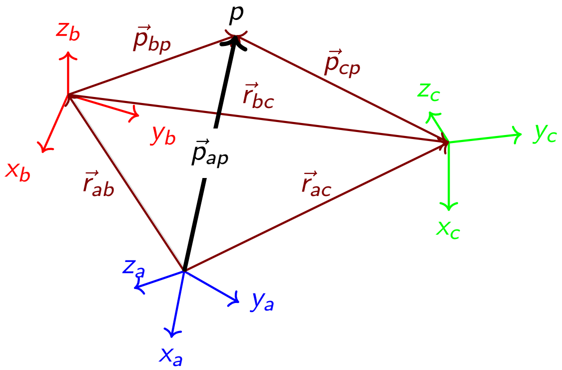
- Recoordinatizing: $\vec{p}^{\,c}_{ap} = C^c_a \vec{p}^{\,a}_{ap}$
  (only frame of reference changes)
- Location *wrt* different frame: $\vec{p}^{\,c}_{cp} = \vec{r}^{\,c}_{cb} + C^c_b \vec{r}^{\,b}_{ba} + C^c_a \vec{p}^{\,a}_{ap}$
  (vector addition in same frame)
  $\neq C^c_a \vec{p}^{\,a}_{ap}$

Determine location of point p from frame $\{c\}$;
$\Rightarrow$ looking for

Translation (more than two coordinate frames)

Determine location of point p from frame $\{c\}$;

$\Rightarrow$ looking for $\vec{p}_{cp}$



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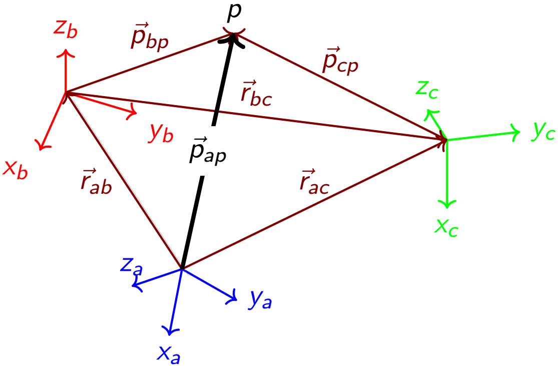

Many approaches given labeled vectors/translations.
$\vec{p}_{cp}$

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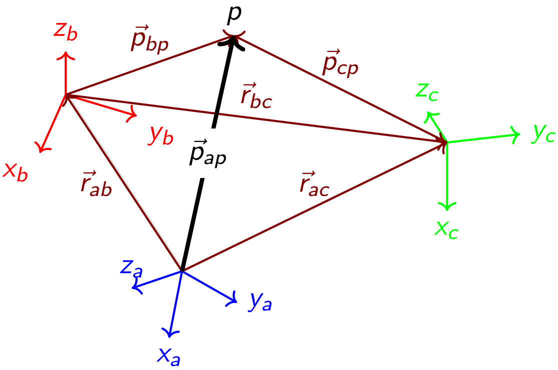

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$$\vec{p}_{cp} = -\vec{r}_{bc} + \vec{p}_{bp}$$

Determine location of point p from frame $\{c\}$;
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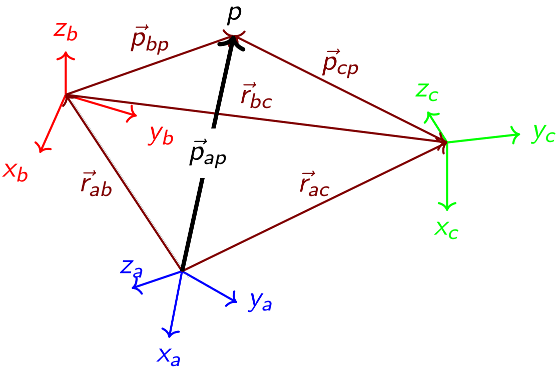

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$$
\begin{aligned}
\vec{p}_{cp} &= -\vec{r}_{bc} + \vec{p}_{bp} \\
&= -\vec{r}_{ac} + \vec{r}_{ab} + \vec{p}_{bp}
\end{aligned}
$$

Determine location of point p from frame $\{c\}$;
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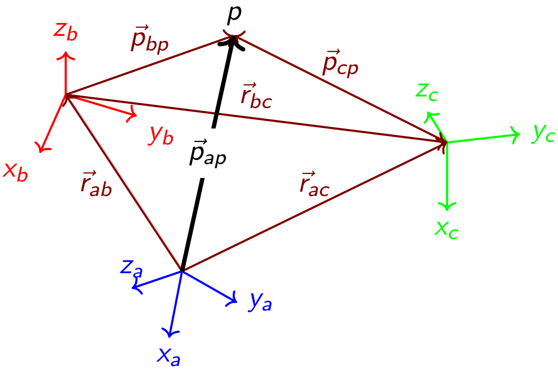

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$$\begin{aligned}\vec{p}_{cp} \\ &= -\vec{r}_{bc} + \vec{p}_{bp} \\ &= -\vec{r}_{ac} + \vec{r}_{ab} + \vec{p}_{bp} \\ &= -\vec{r}_{ac} + \vec{p}_{ap}\end{aligned}$$

Determine location of point p from frame $\{c\}$;
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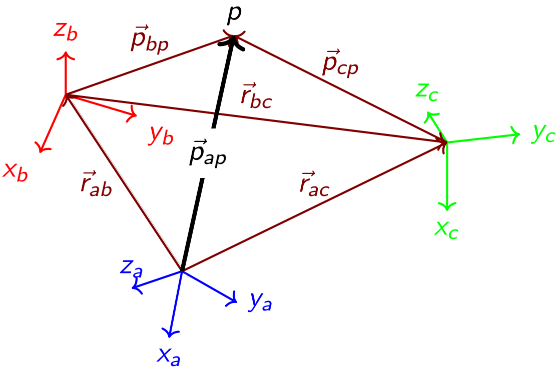

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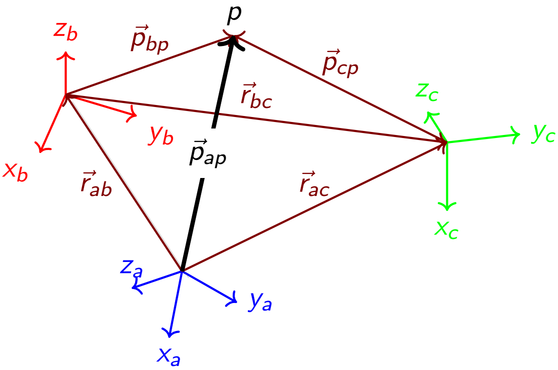
$$\vec{p}_{cp} = -\vec{r}_{bc} + \vec{p}_{bp} = -\vec{r}_{ac} + \vec{r}_{ab} + \vec{p}_{bp} = -\vec{r}_{ac} + \vec{p}_{ap}$$

- In what frame?

Determine location of point p from frame $\{c\}$;
$\Rightarrow$ looking for $\vec{p}_{cp}$



Many approaches given labeled vectors/translations.

$$\vec{p}_{cp} = -\vec{r}_{bc} + \vec{p}_{bp} = -\vec{r}_{ac} + \vec{r}_{ab} + \vec{p}_{bp} = -\vec{r}_{ac} + \vec{p}_{ap}$$

- In what frame? doesn't matter, so long as same
- Can always recoordinatize given C^a_b, C^b_c, C^c_a

Example

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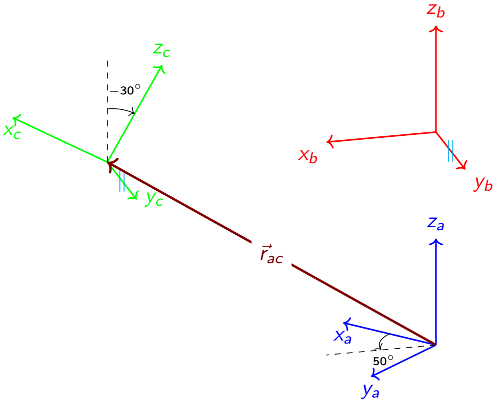
Consider the three coordinate frames $\{a\}, \{b\}, \{c\}$ shown with the rotations and translations between some frames given.



$$C_b^a = R_{z,50^\circ}$$
$$C_c^b = R_{y,-30^\circ}$$
$$\vec{r}^{\,a}_{ab} = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix}^T$$
$$\vec{r}^{\,b}_{bc} = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}^T$$

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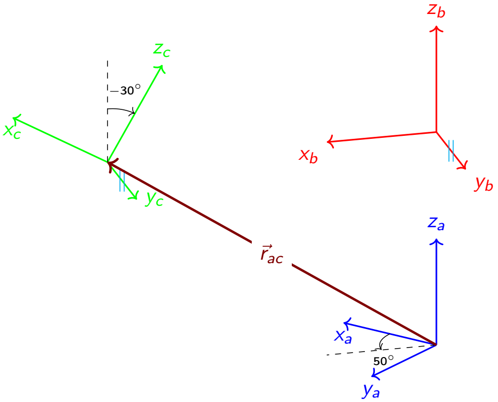
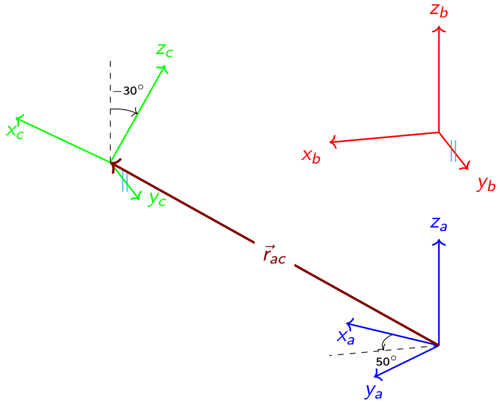


$$C_b^a = R_{z,50^\circ}$$
$$C_c^b = R_{y,-30^\circ}$$
$$\vec{r}_{ab}^{\,a} = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix}^T$$
$$\vec{r}_{bc}^{\,b} = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}^T$$

- find

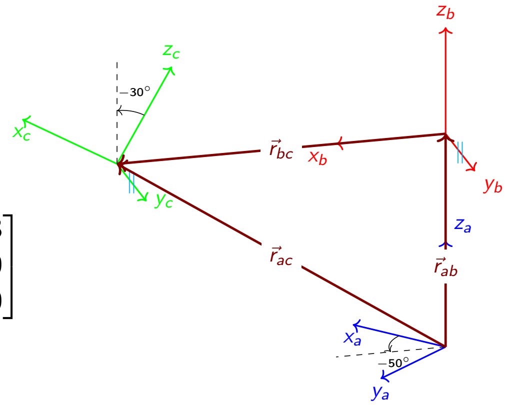
$$C_c^a$$
$$\vec{r}_{ac}^{\,a}$$
$$\vec{r}_{ca}^{\,c}$$

$$C_c^a = C_b^a C_c^b = R_{z,50^\circ} R_{y,-30^\circ}$$



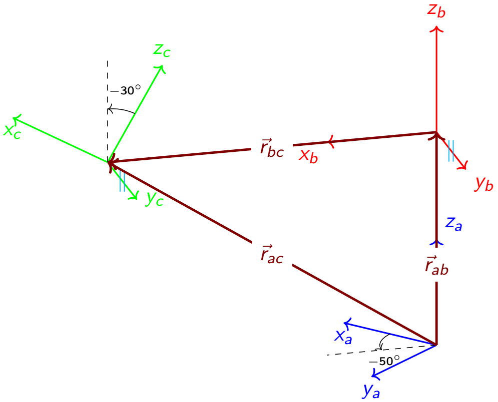
Example - Find $\vec{r}^{\,a}_{ac}$

$$
\begin{aligned}
\vec{r}^{\,a}_{ac} &= \vec{r}^{\,a}_{ab} + \vec{r}^{\,a}_{bc} \\
&= \vec{r}^{\,a}_{ab} + C^a_b \vec{r}^{\,b}_{bc} \\
&= \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + R_{z,50^\circ} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} \cos 50^\circ & -\sin 50^\circ & 0 \\ \sin 50^\circ & \cos 50^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1.93 \\ 2.30 \\ 2.00 \end{bmatrix}
\end{aligned}
$$



Example - Find $\vec{r}^{\,c}_{ca}$

$$
\begin{aligned}
\vec{r}^{\,c}_{ca} &= -\vec{r}^{\,c}_{ac} \\
&= -C^c_a \vec{r}^{\,a}_{ac} \\
&= -\left[C^a_c\right]^T \vec{r}^{\,a}_{ac} \\
&= -\left[R_{z,50^\circ}\ R_{y,-30^\circ}\right]^T \begin{bmatrix} 1.93 \\ 2.30 \\ 2.00 \end{bmatrix} \\
&= \begin{bmatrix} -3.59 \\ 0 \\ -0.232 \end{bmatrix}
\end{aligned}
$$



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