EE 565: Position, Navigation and Timing Navigation Mathematics: Angular and Linear Velocity

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Introduction to Velocity

- Oerivative of Rotation Matrix and Angular Velocity Approach I
- Oerivative of Rotation Matrix and Angular Velocity Approach II
- Properties of Skew-symmetric Matrices
- 6 Propagation/Addition of Angular Velocity
- Iinear Position, Velocity and Acceleration

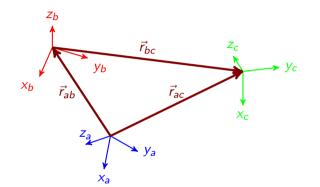
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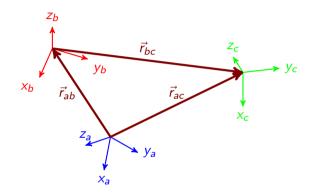


 translation between frames {a} and {c}:

$$\vec{r}_{ac} = \vec{r}_{ab} + \vec{r}_{bc}$$







• translation between frames {*a*} and {*c*}:

$$\vec{r}_{ac} = \vec{r}_{ab} + \vec{r}_{bc}$$

• written *wrt*/frame {*a*}

$$\vec{r}_{ac}^{a} = \vec{r}_{ab}^{a} + \vec{r}_{bc}^{a}$$
$$= \vec{r}_{ab}^{a} + C_{b}^{a}\vec{r}_{bc}^{b}$$

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Intro to Vel

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$$\vec{r}_{ac}^{a}=\vec{r}_{ab}^{a}+C_{b}^{a}\vec{r}_{bc}^{b}$$

what is linear velocity between frames?





$$ec{r}^{a}_{ac}=ec{r}^{a}_{ab}+C^{a}_{b}ec{r}^{b}_{bc}$$

what is linear velocity between frames?

$$\begin{aligned} \vec{r}^{a}_{ac} &\equiv \frac{d}{dt} \vec{r}^{a}_{ac} \\ &= \frac{d}{dt} \left(\vec{r}^{a}_{ab} + C^{a}_{b} \vec{r}^{b}_{bc} \right) \\ &= \vec{r}^{a}_{ab} + C^{a}_{b} \vec{r}^{b}_{bc} + C^{a}_{b} \vec{r}^{b}_{bc} \end{aligned}$$

• Why is $\dot{C}_b^a \neq 0$ in general?





$$ec{r}^{a}_{ac}=ec{r}^{a}_{ab}+C^{a}_{b}ec{r}^{b}_{bc}$$

what is linear velocity between frames?

• Why is $\dot{C}_b^a \neq 0$ in general? Recoordinatization of \vec{r}_{bc}^{b} is time-dependent.





$$\vec{r}^{\,a}_{\,ac}=\vec{r}^{\,a}_{\,ab}+C^a_b\vec{r}^{\,b}_{\,bc}$$

what is linear velocity between frames?

$$\begin{aligned} \dot{\vec{r}}^{a}_{ac} &\equiv \frac{d}{dt}\vec{r}^{a}_{ac} \\ &= \frac{d}{dt}\left(\vec{r}^{a}_{ab} + C^{a}_{b}\vec{r}^{b}_{bc}\right) \\ &= \dot{\vec{r}}^{a}_{ab} + \dot{C}^{a}_{b}\vec{r}^{b}_{bc} + C^{a}_{b}\dot{\vec{r}}^{b}_{bc} \end{aligned}$$

- Why is $\dot{C}_b^a \neq 0$ in general? Recoordinatization of \vec{r}_{bc}^{b} is time-dependent.
- \dot{C}_b^a is directly related to angular velocity between frames $\{a\}$ and $\{b\}$.



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Given a rotation matrix C, one of its properties is

$$[C_b^a]^T C_b^a = C_b^a [C_b^a]^T = \mathcal{I}$$

Taking the time-derivative of the "right-inverse" property

$$\frac{d}{dt}\left(C_{b}^{a}[C_{b}^{a}]^{T}\right) = \frac{d}{dt}\mathcal{I}$$





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$$\Rightarrow \underbrace{\dot{C}_{b}^{a}[C_{b}^{a}]^{T}}_{\Omega_{ab}^{a}} + \underbrace{\underbrace{C_{b}^{a}[\dot{C}_{b}^{a}]^{T}}_{(\dot{C}_{b}^{a}[C_{b}^{a}]^{T})^{T}} = 0$$

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$$\underbrace{(\dot{C}_{b}^{a}[C_{b}^{a}]^{T})^{T}}_{[\Omega_{ab}^{a}]^{T}} = 0$$

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Given a rotation matrix C, one of its properties is

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Taking the time-derivative of the "right-inverse" property

$$\frac{d}{dt}\left(C_b^a[C_b^a]^T\right) = \frac{d}{dt}\mathcal{I}$$

$$\Rightarrow \underbrace{\dot{C}_{b}^{a}[C_{b}^{a}]^{T}}_{\Omega_{ab}^{a}} + \underbrace{\underbrace{C_{b}^{a}[\dot{C}_{b}^{a}]^{T}}_{(\dot{C}_{b}^{a}[C_{b}^{a}]^{T})^{T}} = 0$$

$$\Rightarrow \Omega^a_{ab} + [\Omega^a_{ab}]^T = 0$$

$\Rightarrow \Omega^a_{ab}$ is skew-symmetric!

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 $\frac{-}{dt}C$ and $\omega = 00$

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Define this skew-symmetric matrix Ω_{ab}^{a}

$$\Omega^{a}_{ab} = \begin{bmatrix} \vec{\omega} \ ^{a}_{ab} \times \end{bmatrix} = \begin{bmatrix} 0 & -\omega_{z} & \omega_{y} \\ \omega_{z} & 0 & -\omega_{x} \\ -\omega_{y} & \omega_{x} & 0 \end{bmatrix}$$

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Define this skew-symmetric matrix Ω^a_{ab}

$$\Omega^{a}_{ab} = \begin{bmatrix} \vec{\omega} \ ^{a}_{ab} \times \end{bmatrix} = \begin{bmatrix} 0 & -\omega_{z} & \omega_{y} \\ \omega_{z} & 0 & -\omega_{x} \\ -\omega_{y} & \omega_{x} & 0 \end{bmatrix}$$

Note
$$\Omega^a_{ab} = \dot{C}^a_b [C^a_b]^T$$

 $\Rightarrow \dot{C}^a_b = \Omega^a_{ab} C^a_b$

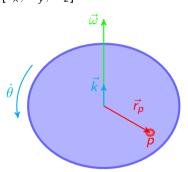
is a means of finding derivative of rotation matrix provided we can further understand Ω^a_{ab} .





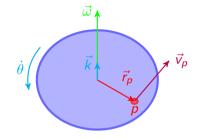
Now for some insight into physical meaning of Ω^a_{ab} .

• Consider a point p on a rigid body rotating with angular velocity $\vec{\omega} = [\omega_x, \omega_v, \omega_z]^T = \dot{\theta}\vec{k} = \dot{\theta}[k_x, k_v, k_z]^T$ with \vec{k} a unit vector.



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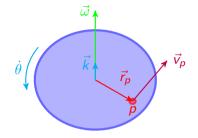




From mechanics, linear velocity $\vec{v_p}$ of point is







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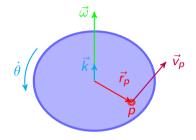
$$\vec{v}_{p} = \vec{\omega} \times \vec{r}_{p} = \begin{bmatrix} \omega_{x} \\ \omega_{y} \\ \omega_{z} \end{bmatrix} \times \begin{bmatrix} r_{x} \\ r_{y} \\ r_{z} \end{bmatrix} = \begin{bmatrix} \omega_{y} r_{z} - \omega_{z} r_{y} \\ \omega_{z} r_{x} - \omega_{x} r_{z} \\ \omega_{x} r_{y} - \omega_{y} r_{x} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\omega_{z} & \omega_{y} \\ \omega_{z} & 0 & -\omega_{x} \\ -\omega_{y} & \omega_{x} & 0 \end{bmatrix}}_{?} \begin{bmatrix} r_{x} \\ r_{y} \\ r_{z} \end{bmatrix}$$

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From mechanics, linear velocity $\vec{v_p}$ of point is

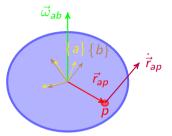
$$\vec{v_{p}} = \vec{\omega} \times \vec{r_{p}} = \begin{bmatrix} \omega_{x} \\ \omega_{y} \\ \omega_{z} \end{bmatrix} \times \begin{bmatrix} r_{x} \\ r_{y} \\ r_{z} \end{bmatrix} = \begin{bmatrix} \omega_{y} r_{z} - \omega_{z} r_{y} \\ \omega_{z} r_{x} - \omega_{x} r_{z} \\ \omega_{x} r_{y} - \omega_{y} r_{x} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\omega_{z} & \omega_{y} \\ \omega_{z} & 0 & -\omega_{x} \\ -\omega_{y} & \omega_{x} & 0 \end{bmatrix}}_{\Omega = [\vec{\omega} \times]} \begin{bmatrix} r_{x} \\ r_{y} \\ r_{z} \end{bmatrix}$$

 $\Rightarrow \Omega$ represents angular velocity and performs cross product

		<u>d</u> /dt C and ω - I 00000000				
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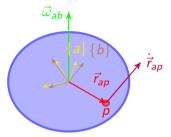
Now let's add fixed frame $\{a\}$ and rotating frame $\{b\}$ attached to moving body such that there is angular velocity $\vec{\omega}_{ab}$ between them.



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Now let's add fixed frame $\{a\}$ and rotating frame $\{b\}$ attached to moving body such that there is angular velocity $\vec{\omega}_{ab}$ between them.



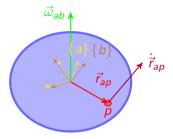
Start with position

$$\vec{r}_{ap}^{a} = \underbrace{\vec{r}_{ab}^{a}}_{0} + C_{b}^{a}\vec{r}_{bp}^{b}$$





Now let's add fixed frame $\{a\}$ and rotating frame $\{b\}$ attached to moving body such that there is angular velocity $\vec{\omega}_{ab}$ between them.



and take derivative wrt time

$$\begin{split} \dot{\vec{r}}^{a}_{ap} &= \underbrace{\dot{C}^{a}_{b}}_{\Omega^{a}_{ab}C^{a}_{b}} \vec{r}^{b}_{bp} + \underbrace{C^{a}_{b}\vec{r}^{b}_{bp}}_{0} \\ &= \Omega^{a}_{ab}C^{a}_{b}\vec{r}^{b}_{bp} \\ &= \Omega^{a}_{ab}\vec{r}^{a}_{bp} = [\vec{\omega}^{a}_{ab}\times]\vec{r}^{a}_{bp} \end{split}$$

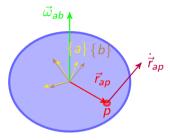
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Now let's add fixed frame $\{a\}$ and rotating frame $\{b\}$ attached to moving body such that there is angular velocity $\vec{\omega}_{ab}$ between them.



Start with position

$$\vec{r}_{ap}^{a} = \underbrace{\vec{r}_{ab}^{a}}_{0} + C_{b}^{a} \vec{r}_{bp}^{b}$$

and take derivative wrt time

$$\begin{array}{ll} \vec{r}^{a}_{ap} & = & \underbrace{\dot{C}^{a}_{b}}_{\Omega^{a}_{ab}C^{a}_{b}} \vec{r}^{\,b}_{\,bp} + \underbrace{C^{a}_{b} \vec{r}^{b}_{bp}}_{0} \\ & = & \Omega^{a}_{ab}C^{a}_{b} \vec{r}^{\,b}_{\,bp} \\ & = & \Omega^{a}_{ab} \vec{r}^{\,a}_{\,bp} = [\vec{\omega}^{\,a}_{\,ab} \times] \vec{r}^{\,a}_{\,bp} \end{array}$$

from which it is observed (compare to $\vec{v_p} = \vec{\omega} \times \vec{r_p}$) that Ω^a_{ab} represents cross product with angular velocity $\vec{\omega}^a_{ab}$.



$$\frac{d}{dt}C$$
 and ω – II

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- Another approach to developing derivative of rotation matrix and angular velocity is based upon angle-axis representation of orientation and rotation matrix as exponential.
- This approach is included in notes.



Properties of SS Matrices

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$$C\Omega C^{\mathsf{T}} \vec{b} = C \left[\vec{\omega} \times \left(C^{\mathsf{T}} \vec{b} \right) \right]$$
$$= C \vec{\omega} \times \left(C C^{\mathsf{T}} \vec{b} \right)$$
$$= C \vec{\omega} \times \vec{b}$$
$$= [C \vec{\omega} \times] \vec{b}$$

Therefore (from above),

$$C\Omega C^{\mathsf{T}} = C[\vec{\omega} \times] C^{\mathsf{T}} = [C\vec{\omega} \times]$$

and (via distributive property)

$$C[\vec{\omega} \times] = [C\vec{\omega} \times]C$$

noting both $\vec{\omega}$ and vector with which cross-product will be taken are assumed to be in the same coordinate frame and thus both need to be recoordinatized.

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Properties of Skew-symmetric Matrices



$$\begin{split} \dot{C}_b^a &= \Omega^a_{ab} C^a_b \\ &= [\vec{\omega}^{\,a}_{\,ab} \times] C^a_b \\ &= [C^a_b \vec{\omega}^{\,b}_{\,ab} \times] C^a_b \\ &= C^a_b [\vec{\omega}^{\,b}_{\,ab} \times] \\ &= C^a_b [\vec{\omega}^{\,b}_{\,ab} \times] \\ &= C^a_b \Omega^b_{ab} \end{split}$$

$$\Rightarrow \dot{C}^a_b = \Omega^a_{ab} C^a_b = C^a_b \Omega^b_{ab}$$





Angular velocity can be

- described as a vector
 - the angular velocity of the *b*-frame wrt the *a*-frame resolved in the *c*-frame, $\vec{\omega}_{ab}^{c}$

•
$$\vec{\omega}_{ab} = -\vec{\omega}_{ba}$$





Angular velocity can be

- described as a vector
 - the angular velocity of the *b*-frame wrt the *a*-frame resolved in the *c*-frame, $\vec{\omega}_{ab}^{c}$
 - $\vec{\omega}_{ab} = -\vec{\omega}_{ba}$
- described as a skew-symmetric matrix $\Omega_{ab}^{c} = [\vec{\omega} \, {}^{c}_{ab} \times]$
 - the skew-symmetric matrix is equivalent to the vector cross product when pre-multiplying another vector





Angular velocity can be

- described as a vector
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 - $\vec{\omega}_{ab} = -\vec{\omega}_{ba}$
- described as a skew-symmetric matrix $\Omega^{c}_{ab} = [\vec{\omega} \, {}^{c}_{ab} \times]$
 - the skew-symmetric matrix is equivalent to the vector cross product when pre-multiplying another vector
- related to the derivative of the rotation matrix

$$\begin{split} \dot{C}^a_b &= \Omega^a_{ab} C^a_b = C^a_b \Omega^b_{ab} \\ \dot{C}^a_b &= -\Omega^a_{ba} C^a_b = -C^a_b \Omega^b_{ba} \end{split}$$

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Add Angular Velocity

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Consider the derivative of the composition of rotations $C_2^0 = C_1^0 C_2^1$.

$$\begin{aligned} \frac{d}{dt}C_{2}^{0} &= \frac{d}{dt}C_{1}^{0}C_{2}^{1} \\ \Rightarrow & \dot{C}_{2}^{0} &= \dot{C}_{1}^{0}C_{2}^{1} + C_{1}^{0}\dot{C}_{2}^{1} \\ \Rightarrow & \Omega_{02}^{0}C_{2}^{0} &= \Omega_{01}^{0}C_{1}^{0}C_{2}^{1} + C_{1}^{0}C_{2}^{1}\Omega_{12}^{2} \\ \Rightarrow & \Omega_{02}^{0} &= \Omega_{01}^{0}C_{2}^{0}\left[C_{2}^{0}\right]^{T} + C_{2}^{0}\Omega_{12}^{2}\left[C_{2}^{0}\right]^{T} \\ \Rightarrow & \left[\vec{\omega}_{02}^{0}\times\right] &= \left[\vec{\omega}_{01}^{0}\times\right] + \left[C_{2}^{0}\vec{\omega}_{12}^{2}\times\right] \\ \Rightarrow & \vec{\omega}_{02}^{0} &= \vec{\omega}_{01}^{0} + \vec{\omega}_{12}^{0} \end{aligned}$$

 \Rightarrow angular velocities (as vectors) add so long as resolved common coordinate system



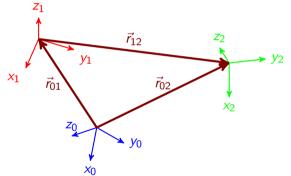
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Linear Position

We can get back to where we started ... motion (translation and rotation) between frames and their derivatives.



Translation (position) between frames {0} and {1}:

$$\vec{r}_{02}^{0} = \vec{r}_{01}^{0} + \vec{r}_{12}^{0} \\ = \vec{r}_{01}^{0} + C_{1}^{0}\vec{r}_{12}^{1}$$

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Linear velocity:

$$\begin{split} \dot{\vec{r}}_{02}^{0}(t) &= \frac{d}{dt} \left(\vec{r}_{01}^{0} + C_{1}^{0} \vec{r}_{12}^{1} \right) \\ &= \dot{\vec{r}}_{01}^{0} + \dot{C}_{1}^{0} \vec{r}_{12}^{1} + C_{1}^{0} \dot{\vec{r}}_{12}^{1} \\ &= \dot{\vec{r}}_{01}^{0} + \Omega_{01}^{0} C_{1}^{0} \vec{r}_{12}^{1} + C_{1}^{0} \dot{\vec{r}}_{12}^{1} \\ &= \dot{\vec{r}}_{01}^{0} + [\vec{\omega}_{01}^{0} \times] C_{1}^{0} \vec{r}_{12}^{1} + C_{1}^{0} \dot{\vec{r}}_{12}^{1} \\ &= \dot{\vec{r}}_{01}^{0} + \vec{\omega}_{01}^{0} \times (C_{1}^{0} \vec{r}_{12}^{1}) + C_{1}^{0} \dot{\vec{r}}_{12}^{1} \end{split}$$

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Linear acceleration:

$$\begin{split} \ddot{r}_{02}^{0} &= \frac{d}{dt} \left(\dot{r}_{01}^{0} + \vec{\omega}_{01}^{0} \times \left(C_{1}^{0} \vec{r}_{12}^{1} \right) + C_{1}^{0} \vec{r}_{12}^{1} \right) \\ &= \ddot{r}_{01}^{0} + \dot{\omega}_{01}^{0} \times \left(C_{1}^{0} \vec{r}_{12}^{1} \right) + \vec{\omega}_{01}^{0} \times \left(\dot{c}_{1}^{0} \vec{r}_{12}^{1} \right) + \vec{\omega}_{01}^{0} \times \left(c_{1}^{0} \vec{r}_{12}^{1} \right) + \vec{\omega}_{01}^{0} \times \left(c_{1}^{0} \vec{r}_{12}^{1} \right) + \vec{c}_{1}^{0} \vec{r}_{12}^{1} + C_{1}^{0} \vec{r}_{12}^{1} \end{split}$$

$$= \ddot{\vec{r}_{01}}^{0} + \dot{\vec{\omega}}_{01}^{0} \times \vec{r}_{12}^{0}(t) + \dot{\vec{\omega}}_{01}^{0} \times \left(\ddot{\vec{\omega}}_{01}^{0} \times \vec{r}_{12}^{0}(t) \right) + 2 \vec{\vec{\omega}}_{01}^{0} \times \left(C_{1}^{0} \dot{\vec{r}}_{12}^{1} \right) + C_{1}^{0} \ddot{\vec{r}}_{12}^{1}$$



Review
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