EE 565: Position, Navigation and Timing

Navigation Mathematics: Angular and Linear Velocity

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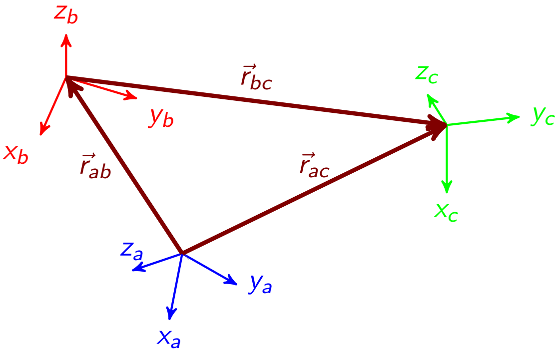
Lecture Topics

Review



- translation between frames $\{a\}$ and $\{c\}$:

$$\vec{r}_{ac} = \vec{r}_{ab} + \vec{r}_{bc}$$

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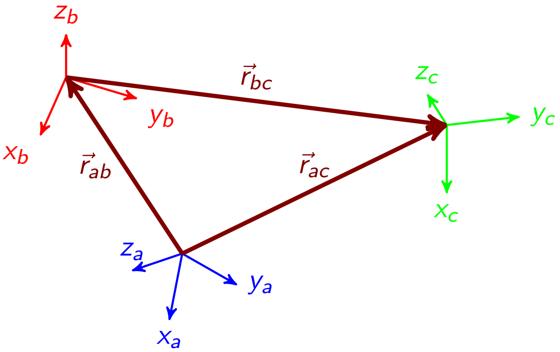

- translation between frames $\{a\}$ and $\{c\}$:

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- written *wrt*/frame $\{a\}$

$$\begin{aligned} \vec{r}^{\,a}_{ac} &= \vec{r}^{\,a}_{ab} + \vec{r}^{\,a}_{bc} \\ &= \vec{r}^{\,a}_{ab} + C^a_b \vec{r}^{\,b}_{bc} \end{aligned}$$

Intro to Vel

- Given relationship for translation between moving (rotating and translating) frames

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what is linear velocity between frames?

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- Why is $\dot{C}^a_b \neq 0$ in general?

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- Why is $\dot{C}^a_b \neq 0$ in general? Recoordinatization of $\vec{r}^{\,b}_{bc}$ is time-dependent.
- $\dot{C}^a_b$ is directly related to angular velocity between frames $\{a\}$ and $\{b\}$.

$\frac{d}{dt}C$ and ω - I

Given a rotation matrix C, one of its properties is

$$[C_b^a]^T C_b^a = C_b^a [C_b^a]^T = \mathcal{I}$$

Taking the time-derivative of the "right-inverse" property

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First approach to $\frac{d}{dt}C$ and angular velocity

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$$\Rightarrow \Omega_{ab}^a + [\Omega_{ab}^a]^T = 0$$

$$\Rightarrow \Omega_{ab}^a \text{ is skew-symmetric!}$$

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Define this skew-symmetric matrix Ω^a_{ab}

$$\Omega^a_{ab} = [\vec{\omega}\,^a_{ab}\times] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

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Note $\Omega^a_{ab} = \dot{C}^a_b[C^a_b]^T$

$$\Rightarrow \dot{C}^a_b = \Omega^a_{ab} C^a_b$$

is a means of finding derivative of rotation matrix provided we can further understand Ω^a_{ab}.

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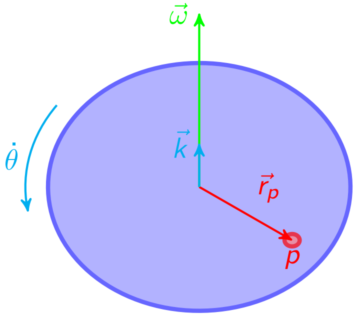
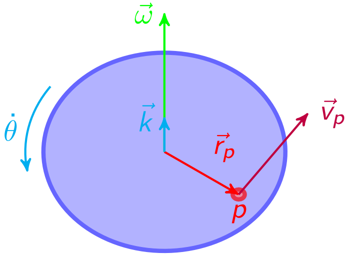
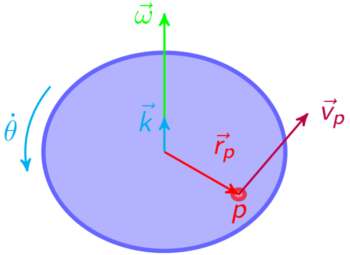
Now for some insight into physical meaning of Ω^a_{ab}.

- Consider a point p on a rigid body rotating with angular velocity $\vec{\omega} = [\omega_x,\ \omega_y,\ \omega_z]^T = \dot{\theta}\vec{k} = \dot{\theta}[k_x,\ k_y,\ k_z]^T$ with $\vec{k}$ a unit vector.



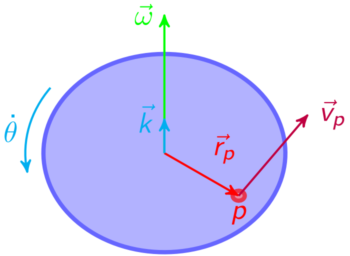


From mechanics, linear velocity $\vec{v}_p$ of point is



From mechanics, linear velocity $\vec{v}_p$ of point is

$$\vec{v}_p = \vec{\omega} \times \vec{r}_p = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \times \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \begin{bmatrix} \omega_y r_z - \omega_z r_y \\ \omega_z r_x - \omega_x r_z \\ \omega_x r_y - \omega_y r_x \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}}_{?} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$



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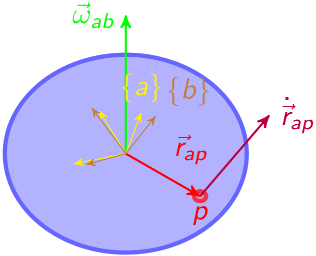
$\Rightarrow \Omega$ represents angular velocity and performs cross product

Now let's add fixed frame $\{a\}$ and rotating frame $\{b\}$ attached to moving body such that there is angular velocity $\vec{\omega}_{ab}$ between them.



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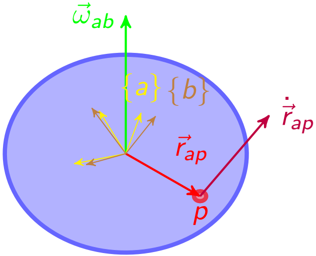

Start with position

$$\vec{r}^{\,a}_{ap} = \underbrace{\vec{r}^{\,a}_{ab}}_{0} + C^a_b \vec{r}^{\,b}_{bp}$$

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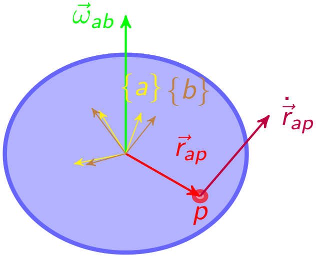

and take derivative *wrt* time

$$\begin{aligned}\dot{\vec{r}}^{\,a}_{ap} &= \underbrace{\dot{C}^a_b}_{\Omega^a_{ab}C^a_b}\,\vec{r}^{\,b}_{bp} + \underbrace{C^a_b\dot{\vec{r}}^{\,b}_{bp}}_{0} \\ &= \Omega^a_{ab}C^a_b\vec{r}^{\,b}_{bp} \\ &= \Omega^a_{ab}\vec{r}^{\,a}_{bp} = [\vec{\omega}^{\,a}_{ab}\times]\vec{r}^{\,a}_{bp}\end{aligned}$$

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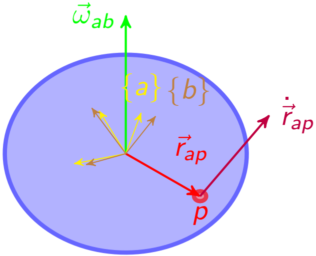

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&= \Omega^a_{ab} C^a_b \vec{r}^{\,b}_{bp} \\
&= \Omega^a_{ab} \vec{r}^{\,a}_{bp} = [\vec{\omega}^{\,a}_{ab} \times] \vec{r}^{\,a}_{bp}
\end{aligned}$$

from which it is observed (compare to $\vec{v}_p = \vec{\omega} \times \vec{r}_p$) that Ω^a_{ab} represents cross product with angular velocity $\vec{\omega}^{\,a}_{ab}$.

$\frac{d}{dt}C$ and ω - II

- Another approach to developing derivative of rotation matrix and angular velocity is based upon angle-axis representation of orientation and rotation matrix as exponential.
- This approach is included in notes.

Properties of SS Matrices

$$
\begin{aligned}
C\Omega C^T\vec{b} &= C\left[\vec{\omega} \times \left(C^T\vec{b}\right)\right] \\
&= C\vec{\omega} \times \left(CC^T\vec{b}\right) \\
&= C\vec{\omega} \times \vec{b} \\
&= [C\vec{\omega}\times]\vec{b}
\end{aligned}
$$

Therefore (from above),

$$C\Omega C^T = C[\vec{\omega}\times]C^T = [C\vec{\omega}\times]$$

and (via distributive property)

$$C[\vec{\omega}\times] = [C\vec{\omega}\times]C$$

noting both $\vec{\omega}$ and vector with which cross-product will be taken are assumed to be in the same coordinate frame and thus both need to be recoordinatized.

$$
\begin{aligned}
\dot{C}_b^a &= \Omega_{ab}^a C_b^a \\
&= [\vec{\omega}\,_{ab}^{a} \times] C_b^a \\
&= [C_b^a \vec{\omega}\,_{ab}^{b} \times] C_b^a \\
&= C_b^a [\vec{\omega}\,_{ab}^{b} \times] \\
&= C_b^a \Omega_{ab}^b
\end{aligned}
$$

$$
\Rightarrow \dot{C}_b^a = \Omega_{ab}^a C_b^a = C_b^a \Omega_{ab}^b
$$

Summary of Angular Velocity and Notation

Angular velocity can be

- described as a vector
  - the angular velocity of the b–frame *wrt* the a–frame resolved in the c–frame, $\vec{\omega}^{\,c}_{ab}$
  - $\vec{\omega}_{ab} = -\vec{\omega}_{ba}$

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- described as a skew-symmetric matrix $\Omega^c_{ab} = [\vec{\omega}^{\,c}_{ab}\times]$
  - the skew-symmetric matrix is equivalent to the vector cross product when pre-multiplying another vector
- related to the derivative of the rotation matrix

$$\dot{C}^a_b = \Omega^a_{ab} C^a_b = C^a_b \Omega^b_{ab}$$
$$\dot{C}^a_b = -\Omega^a_{ba} C^a_b = -C^a_b \Omega^b_{ba}$$

Add Angular Velocity

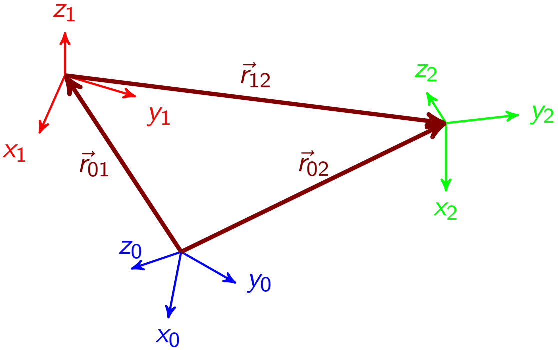
Consider the derivative of the composition of rotations $C_2^0 = C_1^0 C_2^1$.

$$
\begin{aligned}
& \tfrac{d}{dt} C_2^0 &&= \frac{d}{dt} C_1^0 C_2^1 \\
\Rightarrow \quad & \dot{C}_2^0 &&= \dot{C}_1^0 C_2^1 + C_1^0 \dot{C}_2^1 \\
\Rightarrow \quad & \Omega_{02}^0 C_2^0 &&= \Omega_{01}^0 C_1^0 C_2^1 + C_1^0 C_2^1 \Omega_{12}^2 \\
\Rightarrow \quad & \Omega_{02}^0 &&= \Omega_{01}^0 C_2^0 \left[C_2^0\right]^T + C_2^0 \Omega_{12}^2 \left[C_2^0\right]^T \\
\Rightarrow \quad & [\vec{\omega}\,_{02}^0 \times] &&= [\vec{\omega}\,_{01}^0 \times] + [C_2^0 \vec{\omega}\,_{12}^2 \times] \\
\Rightarrow \quad & \vec{\omega}\,_{02}^0 &&= \vec{\omega}\,_{01}^0 + \vec{\omega}\,_{12}^0
\end{aligned}
$$

$\Rightarrow$ angular velocities (as vectors) add so long as resolved common coordinate system

Pos, Vel & Accel

We can get back to where we started ... motion (translation and rotation) between frames and their derivatives.



Translation (position) between frames {0} and {1}:

$$\begin{aligned}\vec{r}^{\,0}_{02} &= \vec{r}^{\,0}_{01} + \vec{r}^{\,0}_{12} \\ &= \vec{r}^{\,0}_{01} + C^0_1 \vec{r}^{\,1}_{12}\end{aligned}$$

Linear velocity:

$$\begin{aligned}
\dot{\vec{r}}^{\,0}_{02}(t) &= \frac{d}{dt}\left(\vec{r}^{\,0}_{01} + C^0_1 \vec{r}^{\,1}_{12}\right) \\
&= \dot{\vec{r}}^{\,0}_{01} + \dot{C}^0_1 \vec{r}^{\,1}_{12} + C^0_1 \dot{\vec{r}}^{\,1}_{12} \\
&= \dot{\vec{r}}^{\,0}_{01} + \Omega^0_{01} C^0_1 \vec{r}^{\,1}_{12} + C^0_1 \dot{\vec{r}}^{\,1}_{12} \\
&= \dot{\vec{r}}^{\,0}_{01} + [\vec{\omega}^{\,0}_{01} \times] C^0_1 \vec{r}^{\,1}_{12} + C^0_1 \dot{\vec{r}}^{\,1}_{12} \\
&= \dot{\vec{r}}^{\,0}_{01} + \vec{\omega}^{\,0}_{01} \times \left(C^0_1 \vec{r}^{\,1}_{12}\right) + C^0_1 \dot{\vec{r}}^{\,1}_{12}
\end{aligned}$$

Linear acceleration:

$$\ddot{\vec{r}}^{\mathbf{0}}_{\mathbf{02}} = \frac{d}{dt}\left(\dot{\vec{r}}^{\mathbf{0}}_{\mathbf{01}} + \vec{\omega}^{\mathbf{0}}_{\mathbf{01}} \times \left(C^{\mathbf{0}}_{\mathbf{1}}\vec{r}^{\mathbf{1}}_{\mathbf{12}}\right) + C^{\mathbf{0}}_{\mathbf{1}}\dot{\vec{r}}^{\mathbf{1}}_{\mathbf{12}}\right)$$

$$= \ddot{\vec{r}}^{\mathbf{0}}_{\mathbf{01}} + \dot{\vec{\omega}}^{\mathbf{0}}_{\mathbf{01}} \times \left(C^{\mathbf{0}}_{\mathbf{1}}\vec{r}^{\mathbf{1}}_{\mathbf{12}}\right) + \vec{\omega}^{\mathbf{0}}_{\mathbf{01}} \times \left(\dot{C}^{\mathbf{0}}_{\mathbf{1}}\vec{r}^{\mathbf{1}}_{\mathbf{12}}\right) + \vec{\omega}^{\mathbf{0}}_{\mathbf{01}} \times \left(C^{\mathbf{0}}_{\mathbf{1}}\dot{\vec{r}}^{\mathbf{1}}_{\mathbf{12}}\right) + \dot{C}^{\mathbf{0}}_{\mathbf{1}}\dot{\vec{r}}^{\mathbf{1}}_{\mathbf{12}} + C^{\mathbf{0}}_{\mathbf{1}}\ddot{\vec{r}}^{\mathbf{1}}_{\mathbf{12}}$$

$$= \ddot{\vec{r}}^{\mathbf{0}}_{\mathbf{01}} + \dot{\vec{\omega}}^{\mathbf{0}}_{\mathbf{01}} \times \vec{r}^{\mathbf{0}}_{\mathbf{12}}(t) + \vec{\omega}^{\mathbf{0}}_{\mathbf{01}} \times \left(\vec{\omega}^{\mathbf{0}}_{\mathbf{01}} \times \vec{r}^{\mathbf{0}}_{\mathbf{12}}(t)\right) + 2\vec{\omega}^{\mathbf{0}}_{\mathbf{01}} \times \left(C^{\mathbf{0}}_{\mathbf{1}}\dot{\vec{r}}^{\mathbf{1}}_{\mathbf{12}}\right) + C^{\mathbf{0}}_{\mathbf{1}}\ddot{\vec{r}}^{\mathbf{1}}_{\mathbf{12}}$$

origin
el ($\omega^2 r$)
2$\omega \times v$)

The End