

# EE 565: Position, Navigation and Timing

## Navigation Mathematics: Kinematics (Angular Velocity: Quaternion Representation)

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- 1 Review
  - Useful Quaternion Properties
  - Angular Velocity
- 2 Angular Velocity Using Quaternions

- DCM (9-elements), e.g.,  $C_1^2 C_0^1$
- Quaternion (4-elements), e.g.,  $\bar{q}_1^2 \otimes \bar{q}_0^1$
- Re-coordinatizing a vector using DCM, e.g.,  $\vec{r}^2 = C_1^2 \vec{r}^1$
- Re-coordinatizing a vector using quaternion, e.g.,  $\vec{r}^2 = \bar{q}_1^2 \otimes \check{r}^1 \otimes (\bar{q}_1^2)^{-1}$ , where  $\check{r} = [0 \quad \vec{r}]^T$

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How many additions and multiplications does each of the above computations require?

- Quaternion multiply

$$\vec{r} = \vec{q} \otimes \vec{p} = [\vec{q} \otimes] \vec{p} = \begin{bmatrix} q_s p_s - \vec{q} \cdot \vec{p} \\ q_s \vec{p} + p_s \vec{q} + \vec{q} \times \vec{p} \end{bmatrix}$$

where

$$[\vec{q} \otimes] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & -q_z & q_y \\ q_y & q_z & q_s & -q_x \\ q_z & -q_y & q_x & q_s \end{bmatrix}$$

- Quaternion multiply (corresponds to reverse order DCM)

$$\vec{r} = \bar{q} \circledast \vec{p} = [\bar{q} \circledast] \vec{p} = \begin{bmatrix} q_s p_s - \vec{q} \cdot \vec{p} \\ q_s \vec{p} + p_s \vec{q} - \vec{q} \times \vec{p} \end{bmatrix}$$

- 

$$\bar{q} \otimes \vec{p} = \vec{p} \circledast \bar{q}$$

where

$$[\bar{q} \circledast] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & q_z & -q_y \\ q_y & -q_z & q_s & q_x \\ q_z & q_y & -q_x & q_s \end{bmatrix}$$

$$\dot{C}_b^a = \Omega_{ab}^a C_b^a \quad (1)$$

where

$$\Omega_{ab}^a = [\vec{\omega}_{ab}^a \times] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

- Since the relative and fixed axis rotations must be performed in a particular order, their derivatives are somewhat challenging
- The angle-axis format, however, is readily differentiable as we can encode the 3 parameters by

$$\vec{K} \equiv \vec{k}(t)\theta(t) = \begin{bmatrix} K_1(t) \\ K_2(t) \\ K_3(t) \end{bmatrix} \quad (2)$$

where  $\theta = \|\vec{K}\|$

- Hence,

$$\frac{d}{dt}\vec{K}(t) = \begin{bmatrix} \dot{K}_1(t) \\ \dot{K}_2(t) \\ \dot{K}_3(t) \end{bmatrix}$$



- For a sufficiently “small” time interval we can often consider the axis of rotation to be  $\approx$  constant (i.e.,  $\vec{K}(t) = \vec{k}$ )

$$\begin{aligned}\frac{d}{dt}\vec{K}(t) &= \frac{d}{dt}(\vec{k}\theta(t)) \\ &= \vec{k}\dot{\theta}(t)\end{aligned}$$

- This is referred to as the angular velocity ( $\vec{\omega}(t)$ ) or the so called “body reference” angular velocity

## Angular Velocity

$$\vec{\omega}(t) \equiv \vec{k}\dot{\theta}(t) \tag{3}$$

- Require minimal storage
- Offer computational advantages over other methods
- Lack of singularities

- Recalling that

$$[\bar{q}_b^a(t) \otimes] = e^{\frac{1}{2}[\check{k}_{ab}^a \otimes]\theta(t)} = \cos(\theta/2)\mathcal{I} + \frac{1}{2}[\check{k}_{ab}^a \otimes] \frac{\sin(\theta/2)}{\theta/2}$$

where

$$\check{k} = \begin{bmatrix} 0 \\ \vec{k} \end{bmatrix}$$

- Hence,

$$\begin{aligned} \frac{d}{dt}[\bar{q}_b^a(t) \otimes] &= \frac{d}{dt} e^{\frac{1}{2}[\check{k}_{ab}^a \otimes]\theta(t)} = \frac{\partial e^{\frac{1}{2}[\check{k}_{ab}^a \otimes]\theta(t)}}{\partial \theta} \frac{d\theta}{dt} \\ &= \frac{1}{2}[\check{k}_{ab}^a \otimes] e^{\frac{1}{2}[\check{k}_{ab}^a \otimes]\theta(t)} \dot{\theta}(t) \\ &= \frac{1}{2} \left( [\check{k}_{ab}^a \otimes] \dot{\theta}(t) \right) [\bar{q}_b^a(t) \otimes] \end{aligned}$$

- let  $W_{ab}^a = \left( [k_{ab}^a] \dot{\theta}(t) \right) = [\check{\omega}_{ab}^a \otimes]$
- therefore,

$$W_{ab}^a = \begin{bmatrix} 0 & -\omega_{ab,x}^a & -\omega_{ab,y}^a & -\omega_{ab,z}^a \\ \omega_{ab,x}^a & 0 & -\omega_{ab,z}^a & \omega_{ab,y}^a \\ \omega_{ab,y}^a & \omega_{ab,z}^a & 0 & -\omega_{ab,x}^a \\ \omega_{ab,z}^a & -\omega_{ab,y}^a & \omega_{ab,x}^a & 0 \end{bmatrix}$$

- and consequently,

$$\dot{\bar{q}}_b^a(t) = \frac{1}{2} [\check{\omega}_{ab}^a \otimes] \bar{q}_b^a(t)$$

- Now,

$$\begin{aligned}\dot{\bar{q}}_b^a(t) &= \frac{1}{2}[\check{\omega}_{ab}^a \otimes] \bar{q}_b^a(t) = \frac{1}{2} \check{\omega}_{ab}^a \otimes \bar{q}_b^a(t) \\ &= \frac{1}{2} \bar{q}_b^a(t) \otimes \check{\omega}_{ab}^b \otimes (\bar{q}_b^a(t))^{-1} \otimes \bar{q}_b^a(t) \\ &= \frac{1}{2} [\bar{q}_b^a(t) \otimes] \check{\omega}_{ab}^b \quad \check{\omega}_{ab}^a = \bar{q}_b^a(t) \otimes \check{\omega}_{ab}^b \otimes (\bar{q}_b^a(t))^{-1} \\ &= \frac{1}{2} [\check{\omega}_{ab}^b \circledast] \bar{q}_b^a(t) \quad (\bar{q}_b^a(t))^{-1} \otimes \bar{q}_b^a(t) = 1\end{aligned}$$

- and consequently,

$$\dot{\bar{q}}_b^a(t) = \frac{1}{2} [\check{\omega}_{ab}^a \otimes] \bar{q}_b^a(t) = \frac{1}{2} [\bar{q}_b^a(t) \otimes] \check{\omega}_{ab}^b = \frac{1}{2} [\check{\omega}_{ab}^b \circledast] \bar{q}_b^a(t) \quad (4)$$