

Lecture

Navigation Equations: ECEF

Mechanization

EE 565: Position, Navigation and Timing

Lecture Notes Update on Spring 2023

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ECEF Mechanization

- Determine the position, velocity and attitude of the **body** frame wrt the **Earth** frame.
 - **Position** — Vector from the origin of the earth frame to the origin of the body frame resolved in the earth frame: \vec{r}_{eb}^e
 - **Velocity** — Velocity of the body frame wrt the earth frame resolved in the earth frame: \vec{v}_{eb}^e
 - **Attitude** — Orientation of the body frame wrt the earth frame: C_b^e

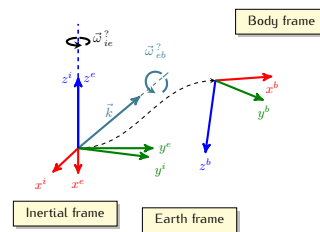
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Attitude — Method A

- Body orientation frame at time “k” wrt time “k – 1”
 - $\Delta t = t_k - t_{k-1}$
- Start with angular velocity

$$\begin{aligned}\vec{\omega}_{ib}^e &= \vec{\omega}_{ie}^e + \vec{\omega}_{eb}^e \\ \vec{\omega}_{eb}^e &= C_b^e \vec{\omega}_{ib}^b - \vec{\omega}_{ie}^e \\ \Omega_{eb}^e &= C_b^e \Omega_{ib}^b C_b^e - \Omega_{ie}^e \\ C_b^e(+)- & C_b^e(-) \approx \Delta t \Omega_{eb}^e C_b^e(-)\end{aligned}$$

$$\begin{aligned}C_b^e(+)& \approx C_b^e(-) + \Delta t (C_b^e \Omega_{ib}^b C_b^e - \Omega_{ie}^e) C_b^e(-) \\ & = C_b^e(-) (\mathcal{I} + \Omega_{ib}^b \Delta t) - \Omega_{ie}^e C_b^e(-) \Delta t\end{aligned}$$



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Attitude — Method B

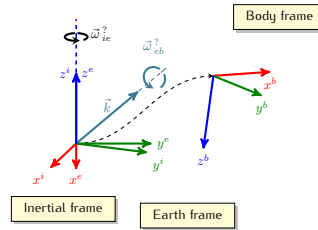
- Body orientation frame at time “k” wrt time “k – 1”
 - $\Delta t = t_k - t_{k-1}$
- Start with the angular velocity

$$\Omega_{eb}^e = C_b^e \Omega_{ib}^b C_e^b - \Omega_{ie}^e$$

$$C_b^e(+)=C_b^e(-)e^{\Omega_{eb}^b \Delta t}=e^{\Omega_{eb}^e \Delta t} C_b^e(-)$$

$$C_b^e(+)=[\mathcal{I}+\sin(\Delta\theta)\mathfrak{K}+[1-\cos(\Delta\theta)]\mathfrak{K}^2]C_b^e(-)$$

$$e^{\Omega_{eb}^e \Delta t}=e^{\mathfrak{K}\theta}$$



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Attitude — Method C

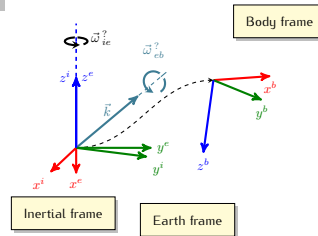
- Body orientation frame at time “k” wrt time “k – 1”
 – $\Delta t = t_k - t_{k-1}$

$$\vec{\omega}_{eb}^e \Delta t = \vec{k} \Delta \theta$$

$$\bar{q}_b^e(+)=\Delta \bar{q}_b^e \otimes \bar{q}_b^e(-)$$

$$\Delta \bar{q}_b^e = \begin{bmatrix} \cos(\frac{\Delta \theta}{2}) \\ \vec{k} \sin(\frac{\Delta \theta}{2}) \end{bmatrix}$$

Need to periodically renormalize \bar{q}



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Attitude Update— Summary

- High fidelity

$$C_b^e(+)=[\mathcal{I}+\sin(\Delta\theta)\mathfrak{K}+[1-\cos(\Delta\theta)]\mathfrak{K}^2]C_b^e(-) \tag{1}$$

or

$$\bar{q}_b^e(+)=\begin{bmatrix} \cos(\frac{\Delta \theta}{2}) \\ \vec{k} \sin(\frac{\Delta \theta}{2}) \end{bmatrix} \otimes \bar{q}_b^e(-) \tag{2}$$

- Low fidelity

$$C_b^e(+)\approx C_b^e(-)(\mathcal{I}+\Omega_{ib}^b \Delta t)-\Omega_{ie}^e C_b^e(-)\Delta t \tag{3}$$

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Steps 2–4

2. Specific force transformation

- Simply coordinatize the specific force

$$\vec{f}_{ib}^e=C_b^e(+)\vec{f}_{ib}^b \tag{4}$$

3. Velocity update

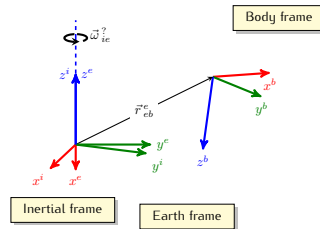
$$\vec{r}_{ib}^i = \cancel{\vec{r}_{ie}^i} + C_e^i \vec{r}_{eb}^e \Rightarrow \vec{r}_{eb}^e = C_i^e \vec{r}_{ib}^i$$

$$\vec{v}_{eb}^e = \dot{\vec{r}}_{eb}^e$$

$$= \dot{C}_i^e \vec{r}_{ib}^i + C_i^e \dot{\vec{r}}_{ib}^i$$

$$= \Omega_{ei}^e C_i^e \vec{r}_{ib}^i + C_i^e \vec{v}_{ib}^i$$

$$= -\Omega_{ie}^e \vec{r}_{eb}^e + C_i^e \vec{v}_{ib}^i$$



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Steps 2-4

$$\vec{a}_{eb}^e = \dot{\vec{v}}_{eb}^e = \frac{d}{dt} (-\Omega_{ie}^e \vec{r}_{eb}^e + C_i^e \vec{v}_{ib}^i)$$

$$= -\Omega_{ie}^e \dot{\vec{r}}_{eb}^e + \dot{C}_i^e \vec{v}_{ib}^i + C_i^e \dot{\vec{v}}_{ib}^i$$

$$= -\Omega_{ie}^e \vec{v}_{eb}^e + \Omega_{ie}^e C_i^e \vec{v}_{ib}^i + C_i^e \vec{a}_{ib}^i$$

$$= -\Omega_{ie}^e \vec{v}_{eb}^e - \Omega_{ie}^e [\vec{v}_{eb}^e + \Omega_{ie}^e \vec{r}_{eb}^e] + C_i^e \vec{a}_{ib}^i$$

$$= -2\Omega_{ie}^e \vec{v}_{eb}^e - \Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^e + \vec{a}_{ib}^e$$

$$= -2\Omega_{ie}^e \vec{v}_{eb}^e + \vec{f}_{ib}^e + \vec{g}_b^e$$

$$C_i^e \vec{v}_{ib}^i = \Omega_{ie}^e \vec{r}_{eb}^e + \vec{v}_{eb}^e$$

$$\vec{f}_{ib}^? = \vec{a}_{ib}^? - \vec{\gamma}_{ib}^?$$

$$\vec{a}_{ib}^e = \vec{f}_{ib}^e + \vec{\gamma}_{ib}^e$$

$$\vec{g}_b^e = \vec{\gamma}_{ib}^e - \Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^e$$

$$\vec{a}_{ib}^e = \vec{f}_{ib}^e + \vec{g}_b^e + \Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^e$$

$$\begin{aligned} \vec{v}_{eb}^e(+) &= \vec{v}_{eb}^e(-) + \vec{a}_{eb}^e \Delta t \\ &= \vec{v}_{eb}^e(-) + \left[\vec{f}_{ib}^e + \vec{g}_b^e - 2\Omega_{ie}^e \vec{v}_{eb}^e(-) \right] \Delta t \end{aligned} \quad (5)$$

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Steps 2-4

4. Position update

- by simple numerical integration

$$\vec{r}_{eb}^e(+) = \vec{r}_{eb}^e(-) + \vec{v}_{eb}^e(-) \Delta t + \vec{a}_{eb}^e \frac{\Delta t^2}{2} \quad (6)$$

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ECEF Mechanization Summary

$$C_b^e(+) = [\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1 - \cos(\Delta\theta)]\mathfrak{K}^2] C_b^e(-)$$

or

$$C_b^e(+) \approx C_b^e(-) (\mathcal{I} + \Omega_{ib}^b \Delta t) - \Omega_{ie}^i C_b^e(-) \Delta t$$

or

$$\vec{q}_b^e(+) = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k} \sin(\frac{\Delta\theta}{2}) \end{bmatrix} \otimes \vec{q}_b^e(-)$$

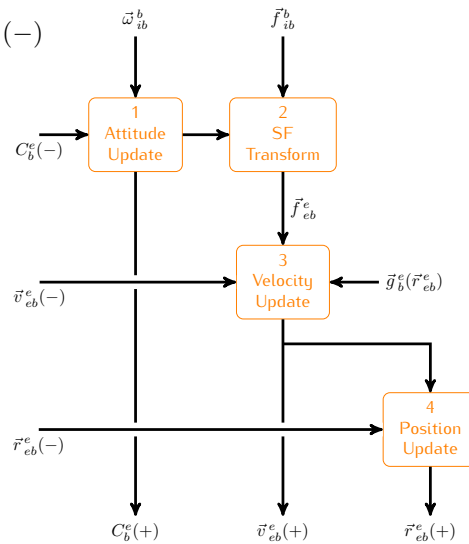
and

$$\vec{f}_{ib}^e = C_b^e(+) \vec{f}_{ib}^b$$

$$\vec{a}_{eb}^e = \vec{f}_{ib}^e + \vec{g}_b^e - 2\Omega_{ie}^i \vec{v}_{eb}^e(-)$$

$$\vec{v}_{eb}^e(+) = \vec{v}_{eb}^e(-) + \vec{a}_{eb}^e \Delta t$$

$$\vec{r}_{eb}^e(+) = \vec{r}_{eb}^e(-) + \vec{v}_{eb}^e(-) \Delta t + \vec{a}_{eb}^e \frac{\Delta t^2}{2}$$



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ECEF Mechanization — Continuous Case

- In continuous time notation
 - Attitude: $\dot{C}_b^e = C_b^e \Omega_{ib}^b$ or $\dot{\vec{q}}_b^e = \frac{1}{2} [\vec{\omega}_{eb}^b \otimes] \vec{q}_b^e(t)$
 - Velocity: $\dot{\vec{v}}_{eb}^e = C_b^e \vec{f}_{ib}^b + \vec{g}_b^e - 2\Omega_{ie}^i \vec{v}_{eb}^e$
 - Position: $\dot{\vec{r}}_{eb}^e = \vec{v}_{eb}^e$

- In State-space notation

$$\begin{bmatrix} \dot{\vec{r}}_{eb}^e \\ \dot{\vec{v}}_{eb}^e \\ \dot{C}_b^i \end{bmatrix} = \begin{bmatrix} \vec{v}_{eb}^e \\ C_b^e \vec{f}_{ib}^b + \vec{g}_b^e - 2\Omega_{ie}^i \vec{v}_{eb}^e \\ C_b^e \Omega_{eb}^b \end{bmatrix} \quad (7)$$

or

$$\begin{bmatrix} \dot{\vec{r}}_{eb}^e \\ \dot{\vec{v}}_{eb}^e \\ \dot{\vec{q}}_b^e \end{bmatrix} = \begin{bmatrix} \vec{v}_{eb}^e \\ C_b^e \vec{f}_{ib}^b + \vec{g}_b^e - 2\Omega_{ie}^i \vec{v}_{eb}^e \\ \frac{1}{2} [\tilde{\omega}_{eb}^b \otimes] \vec{q}_b^e(t) \end{bmatrix} \quad (8)$$

where $\vec{\omega}_{ib}^b = \vec{\omega}_{ie}^b + \vec{\omega}_{eb}^b$, and $\Omega_{eb}^b = \Omega_{ib}^b - \Omega_{ie}^b$.

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Appendix

$$[\vec{q} \otimes] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & -q_z & q_y \\ q_y & q_z & q_s & -q_x \\ q_z & -q_y & q_x & q_s \end{bmatrix}$$

$$[\vec{q} \otimes] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & q_z & -q_y \\ q_y & -q_z & q_s & q_x \\ q_z & q_y & -q_x & q_s \end{bmatrix}$$

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