

EE 565: Position, Navigation and Timing

Navigation Equations: ECEF Mechanization

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- Determine the position, velocity and attitude of the **body** frame *wrt* the **Earth** frame.
 - **Position** — Vector from the origin of the earth frame to the origin of the body frame resolved in the earth frame: \vec{r}_{eb}^e
 - **Velocity** — Velocity of the body frame *wrt* the earth frame resolved in the earth frame: \vec{v}_{eb}^e
 - **Attitude** — Orientation of the body frame *wrt* the earth frame: C_b^e

- Body orientation frame at time “ k ” wrt time “ $k - 1$ ”
 - $\Delta t = t_k - t_{k-1}$
- Start with angular velocity

$$\vec{\omega}_{ib}^e =$$

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- Start with angular velocity

$$\vec{\omega}_{ib}^e = \vec{\omega}_{ie}^e + \vec{\omega}_{eb}^e$$

$$\vec{\omega}_{eb}^e =$$

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$$\vec{\omega}_{ib}^e = \vec{\omega}_{ie}^e + \vec{\omega}_{eb}^e$$
$$\vec{\omega}_{eb}^e = C_b^e \vec{\omega}_{ib}^b - \vec{\omega}_{ie}^e$$

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$$\Omega_{eb}^e = C_b^e \Omega_{ib}^b C_e^b - \Omega_{ie}^e$$

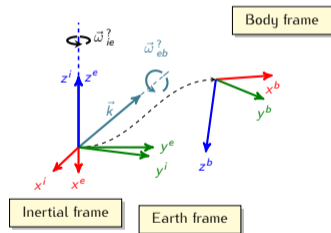
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$$C_b^e(+)- C_b^e(-) \approx$$



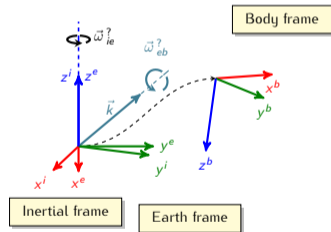
- Body orientation frame at time “ k ” wrt time “ $k - 1$ ”
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$$C_b^e(+)- C_b^e(-) \approx \Delta t \Omega_{eb}^e C_b^e(-)$$



- Body orientation frame at time “ k ” wrt time “ $k - 1$ ”
 - $\Delta t = t_k - t_{k-1}$
- Start with angular velocity

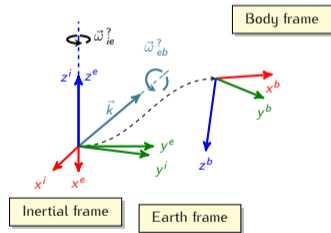
$$\vec{\omega}_{ib}^e = \vec{\omega}_{ie}^e + \vec{\omega}_{eb}^e$$

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$$\Omega_{eb}^e = C_b^e \Omega_{ib}^b C_e^b - \Omega_{ie}^e$$

$$C_b^e(+)- C_b^e(-) \approx \Delta t \Omega_{eb}^e C_b^e(-)$$

$$\begin{aligned} C_b^e(+)&\approx C_b^e(-) + \Delta t \left(C_b^e \Omega_{ib}^b C_e^b - \Omega_{ie}^e \right) C_b^e(-) \\ &= C_b^e(-) \left(\mathcal{I} + \Omega_{ib}^b \Delta t \right) - \Omega_{ie}^e C_b^e(-) \Delta t \end{aligned}$$



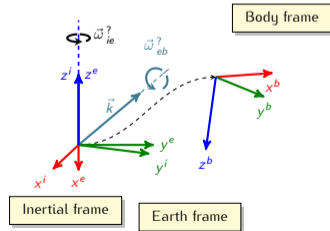
- Body orientation frame at time “ k ” wrt time “ $k - 1$ ”
 - $\Delta t = t_k - t_{k-1}$
- Start with the angular velocity

$$\Omega_{eb}^e = C_b^e \Omega_{ib}^b C_e^b - \Omega_{ie}^e$$

$$C_b^e(+)=C_b^e(-)e^{\Omega_{eb}^b \Delta t}=e^{\Omega_{eb}^e \Delta t} C_b^e(-)$$

$$C_b^e(+)=\left[\mathcal{I}+\sin(\Delta\theta)\hat{\mathcal{K}}+[1-\cos(\Delta\theta)]\hat{\mathcal{K}}^2\right]C_b^e(-)$$

$$e^{\Omega_{eb}^e \Delta t}=e^{\hat{\mathcal{K}}\theta}$$



- Body orientation frame at time “ k ” wrt time “ $k - 1$ ”

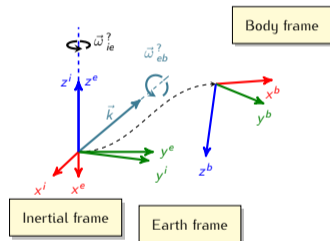
- $\Delta t = t_k - t_{k-1}$

$$\vec{\omega}_{eb}^e \Delta t = \vec{k} \Delta \theta$$

$$\bar{q}_b^e(+)=\Delta \bar{q}_b^e \otimes \bar{q}_b^e(-)$$

$$\Delta \bar{q}_b^e = \begin{bmatrix} \cos\left(\frac{\Delta \theta}{2}\right) \\ \vec{k} \sin\left(\frac{\Delta \theta}{2}\right) \end{bmatrix}$$

Need to periodically renormalize \bar{q}



$$\vec{\omega}_{eb}^e \Delta t = \vec{k} \Delta \theta$$

$$\mathfrak{K} = [\vec{k} \times]$$

- High fidelity

$$C_b^e(+)= [\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1 - \cos(\Delta\theta)] \mathfrak{K}^2] C_b^e(-) \quad (1)$$

or

$$\bar{q}_b^e(+)= \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k} \sin(\frac{\Delta\theta}{2}) \end{bmatrix} \otimes \bar{q}_b^e(-) \quad (2)$$

- Low fidelity

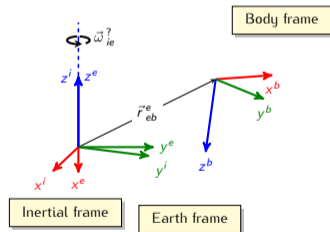
$$C_b^e(+)\approx C_b^e(-) (\mathcal{I} + \Omega_{ib}^b \Delta t) - \Omega_{ie}^e C_b^e(-) \Delta t \quad (3)$$

- 2 Specific force transformation
 - Simply coordinatize the specific force

$$\vec{f}_{ib}^e = C_b^e(+)\vec{f}_{ib}^b \quad (4)$$

- 3 Velocity update

$$\vec{r}_{ib}^i =$$



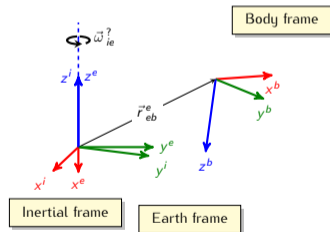
2 Specific force transformation

- Simply coordinatize the specific force

$$\vec{f}_{ib}^e = C_b^e(+)\vec{f}_{ib}^b \quad (4)$$

3 Velocity update

$$\vec{r}_{ib}^i = \cancel{\vec{r}_{ie}^i}^0 + C_e^i \vec{r}_{eb}^e \Rightarrow \vec{r}_{eb}^e = C_i^e \vec{r}_{ib}^i$$



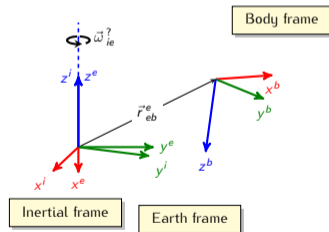
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$$\vec{v}_{eb}^e =$$



2 Specific force transformation

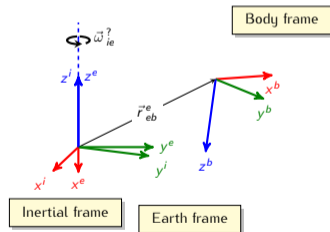
- Simply coordinatize the specific force

$$\vec{f}_{ib}^e = C_b^e(+)\vec{f}_{ib}^b \quad (4)$$

3 Velocity update

$$\vec{r}_{ib}^i = \cancel{\vec{r}_{ie}^i}^0 + C_e^i \vec{r}_{eb}^e \Rightarrow \vec{r}_{eb}^e = C_i^e \vec{r}_{ib}^i$$

$$\begin{aligned} \vec{v}_{eb}^e &= \dot{\vec{r}}_{eb}^e \\ &= \dot{C}_i^e \vec{r}_{ib}^i + C_i^e \dot{\vec{r}}_{ib}^i \\ &= \Omega_{ei}^e C_i^e \vec{r}_{ib}^i + C_i^e \vec{v}_{ib}^i \\ &= -\Omega_{ie}^e \vec{r}_{eb}^e + C_i^e \vec{v}_{ib}^i \end{aligned}$$



$$\begin{aligned}\vec{a}_{eb}^e &= \dot{\vec{v}}_{eb}^e = \frac{d}{dt} (-\Omega_{ie}^e \vec{r}_{eb}^e + C_i^e \vec{v}_{ib}^i) \\ &= -\dot{\Omega}_{ie}^e \vec{r}_{eb}^e + \dot{C}_i^e \vec{v}_{ib}^i + C_i^e \dot{\vec{v}}_{ib}^i \\ &= \end{aligned}$$

$$\begin{aligned}\vec{a}_{eb}^e &= \dot{\vec{v}}_{eb}^e = \frac{d}{dt} (-\Omega_{ie}^e \vec{r}_{eb}^e + C_i^e \vec{v}_{ib}^i) \\ &= -\Omega_{ie}^e \dot{\vec{r}}_{eb}^e + \dot{C}_i^e \vec{v}_{ib}^i + C_i^e \dot{\vec{v}}_{ib}^i \\ &= -\Omega_{ie}^e \vec{v}_{eb}^e + \Omega_{ei}^e C_i^e \vec{v}_{ib}^i + C_i^e \vec{a}_{ib}^i\end{aligned}$$

$$\begin{aligned}
 \vec{a}_{eb}^e &= \dot{\vec{v}}_{eb}^e = \frac{d}{dt} (-\Omega_{ie}^e \vec{r}_{eb}^e + C_i^e \vec{v}_{ib}^i) \\
 &= -\Omega_{ie}^e \dot{\vec{r}}_{eb}^e + \dot{C}_i^e \vec{v}_{ib}^i + C_i^e \dot{\vec{v}}_{ib}^i \\
 &= -\Omega_{ie}^e \vec{v}_{eb}^e + \Omega_{ei}^e C_i^e \vec{v}_{ib}^i + C_i^e \vec{a}_{ib}^i
 \end{aligned}$$

$$C_i^e \vec{v}_{ib}^i = \Omega_{ie}^e \vec{r}_{eb}^e + \vec{v}_{eb}^e$$

$$\begin{aligned}
 \vec{a}_{eb}^e &= \dot{\vec{v}}_{eb}^e = \frac{d}{dt} (-\Omega_{ie}^e \vec{r}_{eb}^e + C_i^e \vec{v}_{ib}^i) \\
 &= -\Omega_{ie}^e \dot{\vec{r}}_{eb}^e + \dot{C}_i^e \vec{v}_{ib}^i + C_i^e \dot{\vec{v}}_{ib}^i \\
 &= -\Omega_{ie}^e \vec{v}_{eb}^e + \Omega_{ei}^e C_i^e \vec{v}_{ib}^i + C_i^e \vec{a}_{ib}^i \\
 &= -\Omega_{ie}^e \vec{v}_{eb}^e - \Omega_{ie}^e [\vec{v}_{eb}^e + \Omega_{ie}^e \vec{r}_{eb}^e] + C_i^e \vec{a}_{ib}^i \\
 &= -2\Omega_{ie}^e \vec{v}_{eb}^e - \Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^e + \vec{a}_{ib}^e
 \end{aligned}$$

$$C_i^e \vec{v}_{ib}^i = \Omega_{ie}^e \vec{r}_{eb}^e + \vec{v}_{eb}^e$$

$$\vec{a}_{eb}^e = \dot{\vec{v}}_{eb}^e = \frac{d}{dt} (-\Omega_{ie}^e \vec{r}_{eb}^e + C_i^e \vec{v}_{ib}^i)$$

$$= -\Omega_{ie}^e \dot{\vec{r}}_{eb}^e + \dot{C}_i^e \vec{v}_{ib}^i + C_i^e \dot{\vec{v}}_{ib}^i$$

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$$= -\Omega_{ie}^e \vec{v}_{eb}^e - \Omega_{ie}^e [\vec{v}_{eb}^e + \Omega_{ie}^e \vec{r}_{eb}^e] + C_i^e \vec{a}_{ib}^i$$

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$$C_i^e \vec{v}_{ib}^i = \Omega_{ie}^e \vec{r}_{eb}^e + \vec{v}_{eb}^e$$

$$\vec{f}_{ib}^? = \vec{a}_{ib}^? - \vec{\gamma}_{ib}^?$$

$$\vec{a}_{ib}^e = \vec{f}_{ib}^e + \vec{\gamma}_{ib}^e$$

$$\vec{g}_b^e = \vec{\gamma}_{ib}^e - \Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^e$$

$$\vec{a}_{ib}^e = \vec{f}_{ib}^e + \vec{g}_b^e + \Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^e$$

$$\vec{a}_{eb}^e = \dot{\vec{v}}_{eb}^e = \frac{d}{dt} (-\Omega_{ie}^e \vec{r}_{eb}^e + C_i^e \vec{v}_{ib}^i)$$

$$= -\Omega_{ie}^e \dot{\vec{r}}_{eb}^e + \dot{C}_i^e \vec{v}_{ib}^i + C_i^e \dot{\vec{v}}_{ib}^i$$

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$$= -2\Omega_{ie}^e \vec{v}_{eb}^e - \Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^e + \vec{a}_{ib}^e$$

$$= -2\Omega_{ie}^e \vec{v}_{eb}^e + \vec{f}_{ib}^e + \vec{g}_b^e$$

$$C_i^e \vec{v}_{ib}^i = \Omega_{ie}^e \vec{r}_{eb}^e + \vec{v}_{eb}^e$$

$$\vec{f}_{ib}^? = \vec{a}_{ib}^? - \vec{\gamma}_{ib}^?$$

$$\vec{a}_{ib}^e = \vec{f}_{ib}^e + \vec{\gamma}_{ib}^e$$

$$\vec{g}_b^e = \vec{\gamma}_{ib}^e - \Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^e$$

$$\vec{a}_{ib}^e = \vec{f}_{ib}^e + \vec{g}_b^e + \Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^e$$

$$\vec{a}_{eb}^e = \dot{\vec{v}}_{eb}^e = \frac{d}{dt} (-\Omega_{ie}^e \vec{r}_{eb}^e + C_i^e \vec{v}_{ib}^i)$$

$$= -\Omega_{ie}^e \dot{\vec{r}}_{eb}^e + \dot{C}_i^e \vec{v}_{ib}^i + C_i^e \dot{\vec{v}}_{ib}^i$$

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$$= -2\Omega_{ie}^e \vec{v}_{eb}^e - \Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^e + \vec{a}_{ib}^e$$

$$= -2\Omega_{ie}^e \vec{v}_{eb}^e + \vec{f}_{ib}^e + \vec{g}_b^e$$

$$\vec{v}_{eb}^e(+)=\vec{v}_{eb}^e(-)+\vec{a}_{eb}^e \Delta t$$

$$= \vec{v}_{eb}^e(-) + \left[\vec{f}_{ib}^e + \vec{g}_b^e - 2\Omega_{ie}^e \vec{v}_{eb}^e(-) \right] \Delta t \quad (5)$$

$$C_i^e \vec{v}_{ib}^i = \Omega_{ie}^e \vec{r}_{eb}^e + \vec{v}_{eb}^e$$

$$\vec{f}_{ib}^? = \vec{a}_{ib}^? - \vec{\gamma}_{ib}^?$$

$$\vec{a}_{ib}^e = \vec{f}_{ib}^e + \vec{\gamma}_{ib}^e$$

$$\vec{g}_b^e = \vec{\gamma}_{ib}^e - \Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^e$$

$$\vec{a}_{ib}^e = \vec{f}_{ib}^e + \vec{g}_b^e + \Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^e$$

- ④ Position update
 - by simple numerical integration

$$\vec{r}_{eb}^e(+)=\vec{r}_{eb}^e(-)+\vec{v}_{eb}^e(-)\Delta t+\vec{a}_{eb}^e\frac{\Delta t^2}{2} \quad (6)$$

$$C_b^e(+)= [\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1 - \cos(\Delta\theta)]\mathfrak{K}^2] C_b^e(-)$$

or

$$C_b^e(+)\approx C_b^e(-)\left(\mathcal{I} + \Omega_{ib}^b\Delta t\right) - \Omega_{ie}^i C_b^e(-)\Delta t$$

or

$$\bar{q}_b^e(+)= \begin{bmatrix} \cos\left(\frac{\Delta\theta}{2}\right) \\ \vec{k}\sin\left(\frac{\Delta\theta}{2}\right) \end{bmatrix} \otimes \bar{q}_b^e(-)$$

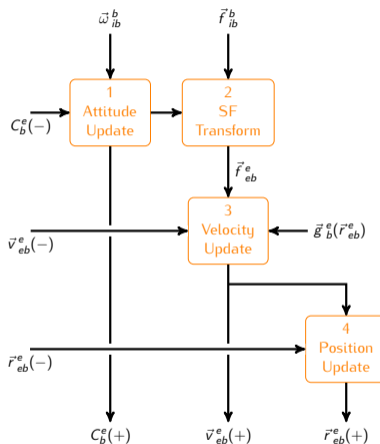
and

$$\vec{f}_{ib}^e = C_b^e(+)\vec{f}_{ib}^b$$

$$\vec{a}_{eb}^e = \vec{f}_{ib}^e + \vec{g}_b^e - 2\Omega_{ie}^i\vec{v}_{eb}^e(-)$$

$$\vec{v}_{eb}^e(+)= \vec{v}_{eb}^e(-) + \vec{a}_{eb}^e\Delta t$$

$$\vec{r}_{eb}^e(+)= \vec{r}_{eb}^e(-) + \vec{v}_{eb}^e(-)\Delta t + \vec{a}_{eb}^e\frac{\Delta t^2}{2}$$



- In continuous time notation

- Attitude: $\dot{C}_b^e = C_b^e \Omega_{eb}^b$ or $\dot{q}_b^e = \frac{1}{2}[\tilde{\omega}_{eb}^b \otimes] \bar{q}_b^e(t)$

$$\vec{\omega}_{ib}^b = \vec{\omega}_{ie}^b + \vec{\omega}_{eb}^b$$

- Velocity: $\dot{\vec{v}}_{eb}^e = C_b^e \vec{f}_{ib}^b + \vec{g}_b^e - 2\Omega_{ie}^i \vec{v}_{eb}^e$

$$\Omega_{eb}^b = \Omega_{ib}^b - \Omega_{ie}^b$$

- Position: $\dot{\vec{r}}_{ib}^e = \vec{v}_{eb}^e$

- In State-space notation

$$\begin{bmatrix} \dot{\vec{r}}_{eb}^e \\ \dot{\vec{v}}_{eb}^e \\ \dot{C}_b^e \end{bmatrix} = \begin{bmatrix} \vec{v}_{eb}^e \\ C_b^e \vec{f}_{ib}^b + \vec{g}_b^e - 2\Omega_{ie}^i \vec{v}_{eb}^e \\ C_b^e \Omega_{eb}^b \end{bmatrix} \quad (7)$$

or

$$\begin{bmatrix} \dot{\vec{r}}_{eb}^e \\ \dot{\vec{v}}_{eb}^e \\ \dot{q}_b^e \end{bmatrix} = \begin{bmatrix} \vec{v}_{eb}^e \\ C_b^e \vec{f}_{ib}^b + \vec{g}_b^e - 2\Omega_{ie}^i \vec{v}_{eb}^e \\ \frac{1}{2}[\tilde{\omega}_{eb}^b \otimes] \bar{q}_b^e(t) \end{bmatrix} \quad (8)$$

$$[\bar{q} \otimes] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & -q_z & q_y \\ q_y & q_z & q_s & -q_x \\ q_z & -q_y & q_x & q_s \end{bmatrix}$$

$$[\bar{q} \circledast] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & q_z & -q_y \\ q_y & -q_z & q_s & q_x \\ q_z & q_y & -q_x & q_s \end{bmatrix}$$