

Lecture

Navigation Equations: ECI Mechanization

EE 565: Position, Navigation and Timing

Lecture Notes Update on Spring 2023

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ECI Mechanization

- Determine the position, velocity and attitude of the **body** frame *wrt* the **inertial** frame
 - **Position** — Vector from the origin of the inertial frame to the origin of the body frame resolved in the inertial frame: \vec{r}_{ib}^i
 - **Velocity** — Velocity of the body frame *wrt* the inertial frame resolved in the inertial frame: \vec{v}_{ib}^i
 - **Attitude** — Orientation of the body frame *wrt* the inertial frame: C_b^i or \bar{q}_b^i
- The inputs are $\vec{\omega}_{ib}^b$ and \vec{f}_{ib}^b

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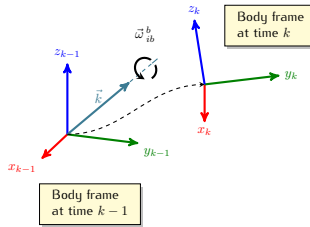
Attitude — Method A

- Body orientation frame at time “k” *wrt* time “k – 1”
 - $\Delta t = t_k - t_{k-1}$

$$\begin{aligned} \dot{C}_b^i &= C_b^i \Omega_{ib}^b \\ &= \lim_{\Delta t \rightarrow 0} \left(\frac{C_b^i(k) - C_b^i(k-1)}{\Delta t} \right) = C_b^i(k-1) \Omega_{ib}^b \end{aligned}$$

$$C_b^i(+)-C_b^i(-) \approx C_b^i(-) \Omega_{ib}^b \Delta t$$

$$C_b^i(+)\approx C_b^i(-) (\mathcal{I} + \Omega_{ib}^b \Delta t)$$



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Attitude — Method B

- Body orientation frame at time “k” *wrt* time “k – 1”
 - $\Delta t = t_k - t_{k-1}$

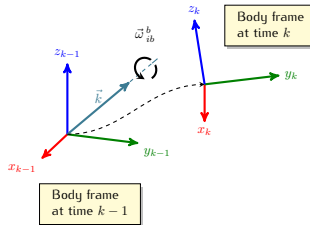
$$\vec{\omega}_{ib}^b \Delta t = \vec{k} \Delta \theta$$

$$C_{b(k)}^i = C_{b(k-1)}^i C_{b(k)}^{b(k-1)}$$

$$\begin{aligned} C_{b(k)}^{b(k-1)} &= e^{\Omega_{ib}^b \Delta t} = e^{\mathfrak{K} \Delta \theta} \\ &= \mathcal{I} + \mathfrak{K} \Delta \theta + \frac{\mathfrak{K}^2 \Delta \theta^2}{2!} + \frac{\mathfrak{K}^3 \Delta \theta^3}{3!} + \dots \\ &= \mathcal{I} + \sin(\Delta \theta) \mathfrak{K} + [1 - \cos(\Delta \theta)] \mathfrak{K}^2 \end{aligned}$$

$$C_b^i(+)=C_b^i(-)C_{b(k)}^{b(k-1)}$$

$$\approx C_b^i(-) (\mathcal{I} + \Omega_{ib}^b \Delta t)$$



$$\mathfrak{K} = [\vec{k} \times]$$

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Attitude — Method C

- Body orientation frame at time “ k ” wrt time “ $k - 1$ ”

$$- \Delta t = t_k - t_{k-1}$$

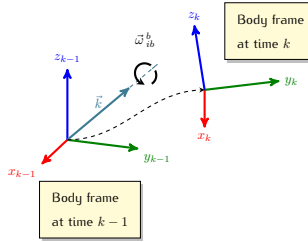
$$\vec{\omega}_{ib}^b \Delta t = \vec{k} \Delta \theta$$

$$\vec{q}_{b(k)}^i = \vec{q}_{b(k-1)}^i \otimes \vec{q}_{b(k)}^{b(k-1)}$$

$$\vec{q}_{b(k)}^{b(k-1)} = \begin{bmatrix} \cos(\frac{\Delta \theta}{2}) \\ \vec{k} \sin(\frac{\Delta \theta}{2}) \end{bmatrix}$$

$$\vec{q}_{b(k)}^i(+) = \vec{q}_{b(k)}^i(-) \otimes \vec{q}_{b(k)}^{b(k-1)}$$

Need to periodically renormalize \vec{q}



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Attitude Update— Summary

- High fidelity

$$C_b^i(+)=C_b^i(-)\left[\mathcal{I}+\sin(\Delta \theta) \mathfrak{K}+[1-\cos(\Delta \theta)] \mathfrak{K}^2\right] \quad (1)$$

or

$$\vec{q}_{b(k)}^i(+)=\vec{q}_{b(k)}^i(-) \otimes \begin{bmatrix} \cos(\frac{\Delta \theta}{2}) \\ \vec{k} \sin(\frac{\Delta \theta}{2}) \end{bmatrix} \quad (2)$$

- Low fidelity

$$C_b^i(+)\approx C_b^i(-)\left(\mathcal{I}+\Omega_{ib}^b \Delta t\right) \quad (3)$$

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Steps 2–4

2. Specific force transformation

- Simply coordinatize the specific force

$$\vec{f}_{ib}^i=C_b^i(+)\vec{f}_{ib}^b \quad (4)$$

3. Velocity update

- Assuming that we are in space (i.e., no centrifugal component)

$$\vec{a}_{ib}^i=\vec{f}_{ib}^i+\vec{\gamma}_{ib}^i \quad (5)$$

- Thus, by simple numerical integration

$$\vec{v}_{ib}^i(+)=\vec{v}_{ib}^i(-)+\vec{a}_{ib}^i \Delta t \quad (6)$$

4. Position update

- by simple numerical integration

$$\vec{r}_{ib}^i(+)=\vec{r}_{ib}^i(-)+\vec{v}_{ib}^i(-) \Delta t+\vec{a}_{ib}^i \frac{\Delta t^2}{2} \quad (7)$$

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ECI Mechanization Summary

$$C_b^i(+)=C_b^i(-)[\mathcal{I}+\sin(\Delta\theta)\mathfrak{K}+[1-\cos(\Delta\theta)]\mathfrak{K}^2]$$

or

$$C_b^i(+)\approx C_b^i(-)(\mathcal{I}+\Omega_{ib}^b\Delta t)$$

or

$$\bar{q}_b^i(+)=\bar{q}_b^i(-)\otimes\begin{bmatrix}\cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2})\end{bmatrix}$$

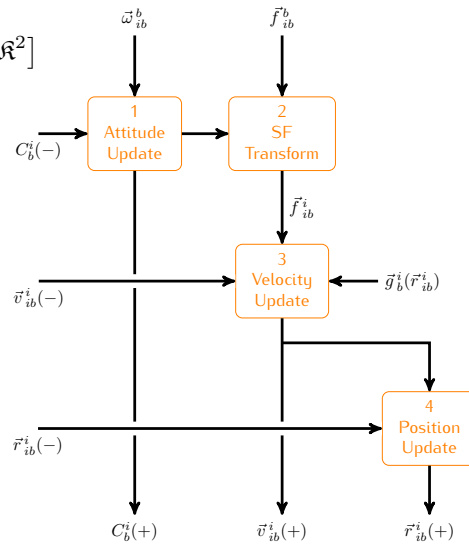
and

$$\vec{f}_{ib}^i=C_b^i(+)\vec{f}_{ib}^b$$

$$\vec{a}_{ib}^i=\vec{f}_{ib}^i+\vec{\gamma}_{ib}^i$$

$$\vec{v}_{ib}^i(+)=\vec{v}_{ib}^i(-)+\vec{a}_{ib}^i\Delta t$$

$$\vec{r}_{ib}^i(+)=\vec{r}_{ib}^i(-)+\vec{v}_{ib}^i(-)\Delta t+\vec{a}_{ib}^i\frac{\Delta t^2}{2}$$



What is the importance of Δt ?

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ECI Mechanization — Continuous Case

- In continuous time notation

– Attitude: $\dot{C}_b^i=C_b^i\Omega_{ib}^b$ or $\dot{\bar{q}}_b^i=\frac{1}{2}[\omega_{ib}^b\otimes]\bar{q}_b^i(t)$

– Velocity: $\dot{\vec{v}}_{ib}^i=C_b^i\vec{f}_{ib}^b+\vec{\gamma}_{ib}^i$

– Position: $\dot{\vec{r}}_{ib}^i=\vec{v}_{ib}^i$

- In State-space notation

$$\begin{bmatrix}\dot{\vec{r}}_{ib}^i \\ \dot{\vec{v}}_{ib}^i \\ \dot{C}_b^i\end{bmatrix}=\begin{bmatrix}\vec{v}_{ib}^i \\ C_b^i\vec{f}_{ib}^b+\vec{\gamma}_{ib}^i \\ C_b^i\Omega_{ib}^b\end{bmatrix}\quad (8)$$

or

$$\begin{bmatrix}\dot{\vec{r}}_{ib}^i \\ \dot{\vec{v}}_{ib}^i \\ \dot{\bar{q}}_b^i\end{bmatrix}=\begin{bmatrix}\vec{v}_{ib}^i \\ C_b^i\vec{f}_{ib}^b+\vec{\gamma}_{ib}^i \\ \frac{1}{2}[\omega_{ib}^b\otimes]\bar{q}_b^i(t)\end{bmatrix}\quad (9)$$

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Appendix

$$[\bar{q}\otimes]=\begin{bmatrix}q_s & -q_x & -q_y & -q_z \\ q_x & q_s & -q_z & q_y \\ q_y & q_z & q_s & -q_x \\ q_z & -q_y & q_x & q_s\end{bmatrix}$$

$$[\bar{q}\otimes]=\begin{bmatrix}q_s & -q_x & -q_y & -q_z \\ q_x & q_s & q_z & -q_y \\ q_y & -q_z & q_s & q_x \\ q_z & q_y & -q_x & q_s\end{bmatrix}$$

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