EE 565: Position, Navigation and Timing

Navigation Equations: ECI Mechanization

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- Determine the position, velocity and attitude of the **body** frame *wrt* the **inertial** frame
  - **Position** — Vector from the origin of the inertial frame to the origin of the body frame resolved in the inertial frame: $\vec{r}^{\,i}_{ib}$
  - **Velocity** — Velocity of the body frame *wrt* the inertial frame resolved in the inertial frame: $\vec{v}^{\,i}_{ib}$
  - **Attitude** — Orientation of the body frame *wrt* the inertial frame: C^i_b or $\bar{q}^{\,i}_b$

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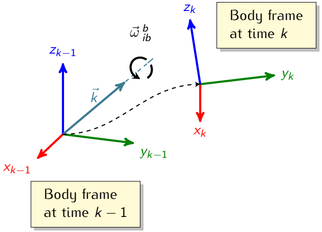
- The inputs are $\vec{\omega}^{\,b}_{ib}$ and $\vec{f}^{\,b}_{ib}$

- Body orientation frame at time "k" *wrt* time "$k-1$"
    - $\Delta t = t_k - t_{k-1}$

$$\dot{C}_b^i = C_b^i \Omega_{ib}^b$$



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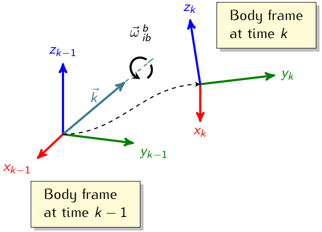
$$= \lim_{\Delta t \to 0} \left(\frac{C_b^i(k) - C_b^i(k-1)}{\Delta t} \right) = C_b^i(k-1)\Omega_{ib}^b$$



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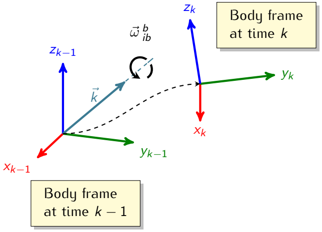
$$C_b^i(+) - C_b^i(-) \approx C_b^i(-)\Omega_{ib}^b \Delta t$$



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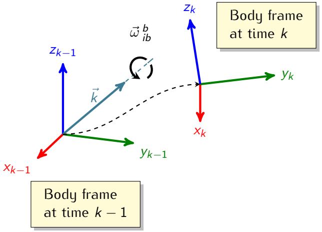

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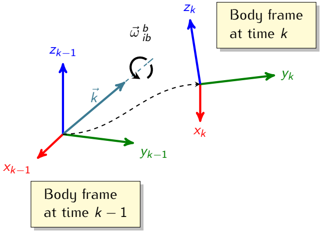
$$C^i_{b(k)} = C^i_{b(k-1)} C^{b(k-1)}_{b(k)}$$

$$\boxed{\vec{\omega}^{\,b}_{ib}\Delta t = \vec{k}\Delta\theta}$$



$$\boxed{\mathfrak{K} = [\vec{k}\times]}$$

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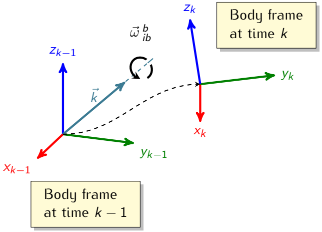

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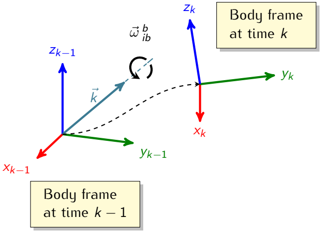

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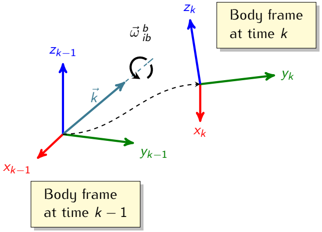

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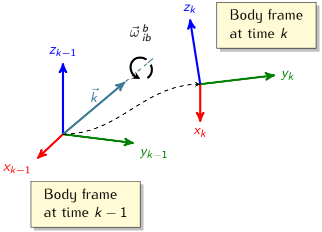

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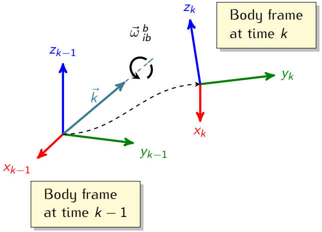

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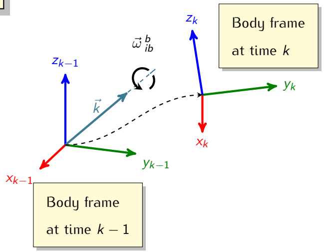
$$\bar{q}^{\,i}_{b(k)} = \bar{q}^{\,i}_{b(k-1)} \otimes \bar{q}^{\,b(k-1)}_{b(k)}$$

$$\bar{q}^{\,b(k-1)}_{b(k)} = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix}$$

$$\bar{q}^{\,i}_{b}(+) = \bar{q}^{\,i}_{b}(-) \otimes \bar{q}^{\,b(k-1)}_{b(k)}$$

Need to periodically renormalize $\bar{q}$



$$\vec{\omega}^{\,b}_{ib}\Delta t = \vec{k}\Delta\theta \qquad \mathfrak{K} = [\vec{k}\times]$$

- High fidelity

$$C^i_b(+) = C^i_b(-)\left[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1-\cos(\Delta\theta)]\,\mathfrak{K}^2\right] \tag{1}$$

or

$$\bar{q}^{\,i}_{\,b}(+) = \bar{q}^{\,i}_{\,b}(-) \otimes \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix} \tag{2}$$

- Low fidelity

$$C^i_b(+) \approx C^i_b(-)\left(\mathcal{I} + \Omega^b_{ib}\Delta t\right) \tag{3}$$

Steps 2–4

2. Specific force transformation
   - Simply coordinatize the specific force

$$\vec{f}^{\,i}_{ib} = C^i_b(+)\vec{f}^{\,b}_{ib} \tag{4}$$

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   - Assuming that we are in space (i.e., no centrifugal component)

$$\vec{f}^{\,i}_{ib} = \vec{a}^{\,i}_{ib} - \vec{\gamma}^{\,i}_{ib}$$

$$\vec{a}^{\,i}_{ib} = \vec{f}^{\,i}_{ib} + \vec{\gamma}^{\,i}_{ib} \tag{5}$$

   - Thus, by simple numerical integration

$$\vec{v}^{\,i}_{ib}(+) = \vec{v}^{\,i}_{ib}(-) + \vec{a}^{\,i}_{ib}\Delta t \tag{6}$$

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   - Thus, by simple numerical integration

$$\vec{v}^{\,i}_{ib}(+) = \vec{v}^{\,i}_{ib}(-) + \vec{a}^{\,i}_{ib}\Delta t \tag{6}$$

4. Position update
   - by simple numerical integration

$$\vec{r}^{\,i}_{ib}(+) = \vec{r}^{\,i}_{ib}(-) + \vec{v}^{\,i}_{ib}(-)\Delta t + \vec{a}^{\,i}_{ib}\frac{\Delta t^2}{2} \tag{7}$$

$$C_b^i(+) = C_b^i(-)\left[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1-\cos(\Delta\theta)]\,\mathfrak{K}^2\right]$$

or

$$C_b^i(+) \approx C_b^i(-)\left(\mathcal{I} + \Omega_{ib}^b \Delta t\right)$$

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$$\vec{a}\,{}^i_{ib} = \vec{f}\,{}^i_{ib} + \vec{\gamma}\,{}^i_{ib}$$
$$\vec{v}\,{}^i_{ib}(+) = \vec{v}\,{}^i_{ib}(-) + \vec{a}\,{}^i_{ib}\Delta t$$
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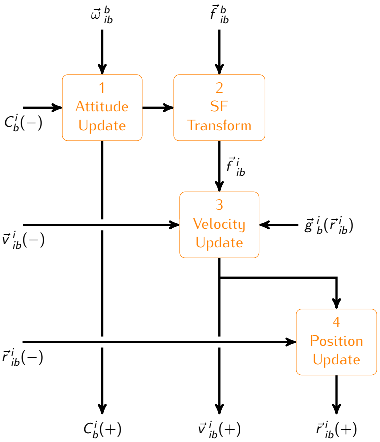

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$$\vec{f}\,{}_{ib}^i = C_b^i(+)\vec{f}\,{}_{ib}^b$$
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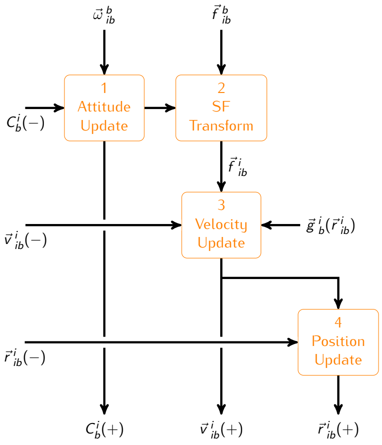

What is the importance of Δt?

- In continuous time notation
  - Attitude: $\dot{C}^i_b = C^i_b \Omega^b_{ib}$ or $\dot{\bar{q}}^i_b = \frac{1}{2}[\breve{\omega}^b_{ib} \circledast]\bar{q}^{\,i}_{\,b}(t)$
  - Velocity: $\dot{\vec{v}}^{\,i}_{\,ib} = C^i_b \vec{f}^{\,b}_{\,ib} + \vec{\gamma}^{\,i}_{\,ib}$
  - Position: $\dot{\vec{r}}^{\,i}_{\,ib} = \vec{v}^{\,i}_{\,ib}$
- In State-space notation

$$\begin{bmatrix} \dot{\vec{r}}^{\,i}_{\,ib} \\ \dot{\vec{v}}^{\,i}_{\,ib} \\ \dot{C}^i_b \end{bmatrix} = \begin{bmatrix} \vec{v}^{\,i}_{\,ib} \\ C^i_b \vec{f}^{\,b}_{\,ib} + \vec{\gamma}^{\,i}_{\,ib} \\ C^i_b \Omega^b_{ib} \end{bmatrix} \tag{8}$$

or

$$\begin{bmatrix} \dot{\vec{r}}^{\,i}_{\,ib} \\ \dot{\vec{v}}^{\,i}_{\,ib} \\ \dot{\bar{q}}^i_b \end{bmatrix} = \begin{bmatrix} \vec{v}^{\,i}_{\,ib} \\ C^i_b \vec{f}^{\,b}_{\,ib} + \vec{\gamma}^{\,i}_{\,ib} \\ \frac{1}{2}[\breve{\omega}^b_{ib} \circledast]\bar{q}^{\,i}_{\,b}(t) \end{bmatrix} \tag{9}$$

$$[\bar{q}\otimes] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & -q_z & q_y \\ q_y & q_z & q_s & -q_x \\ q_z & -q_y & q_x & q_s \end{bmatrix}$$

$$[\bar{q}\circledast] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & q_z & -q_y \\ q_y & -q_z & q_s & q_x \\ q_z & q_y & -q_x & q_s \end{bmatrix}$$