Lecture

Navigation Equations: Nav Mechanization

EE 565: Position, Navigation and Timing

Lecture Notes Update on Spring 2023

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Nav Mechanization

- Determine the position, velocity and attitude of the **body** frame *wrt* the **Nav** frame.
  - **Position** — Typically described in curvlinear coordinates: $[L_b, \lambda_b, h_b]^T$
  - **Velocity** — Velocity of the body frame *wrt* the earth frame resolved in the navigation frame: $\vec{v}^{\,n}_{eb}$
  - **Attitude** — Orientation of the body frame *wrt* the navigation frame: C^n_b

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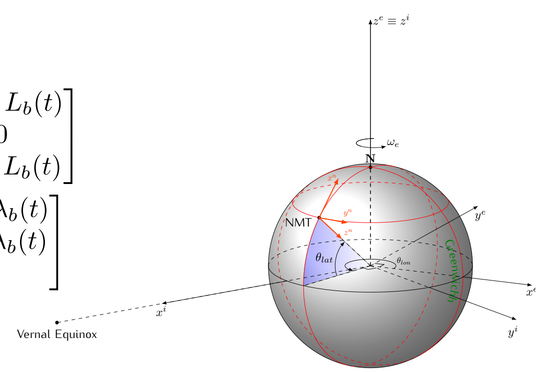
ECEF/Nav

- Description of the Nav frame
  - Orientation of the n-frame *wrt* the e-frame

$$
\begin{aligned}
C^e_n(t) &= R_{(\vec{z},\lambda_b(t))} R_{(\vec{y},-L_b(t)-90^\circ)} \\
&= \begin{bmatrix} \cos\lambda_b(t) & -\sin\lambda_b(t) & 0 \\ \sin\lambda_b(t) & \cos\lambda_b(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin L_b(t) & 0 & -\cos L_b(t) \\ 0 & 1 & 0 \\ \cos L_b(t) & 0 & -\sin L_b(t) \end{bmatrix} \\
&= \begin{bmatrix} -\sin L_b(t)\cos\lambda_b(t) & -\sin\lambda_b(t) & -\cos L_b(t)\cos\lambda_b(t) \\ -\sin L_b(t)\sin\lambda_b(t) & \cos\lambda_b(t) & -\cos L_b(t)\sin\lambda_b(t) \\ \cos L_b(t) & 0 & -\sin L_b(t) \end{bmatrix}
\end{aligned}
$$

where geodetic Lat $= L_b$ and Geodetic Lon $= \lambda_b$



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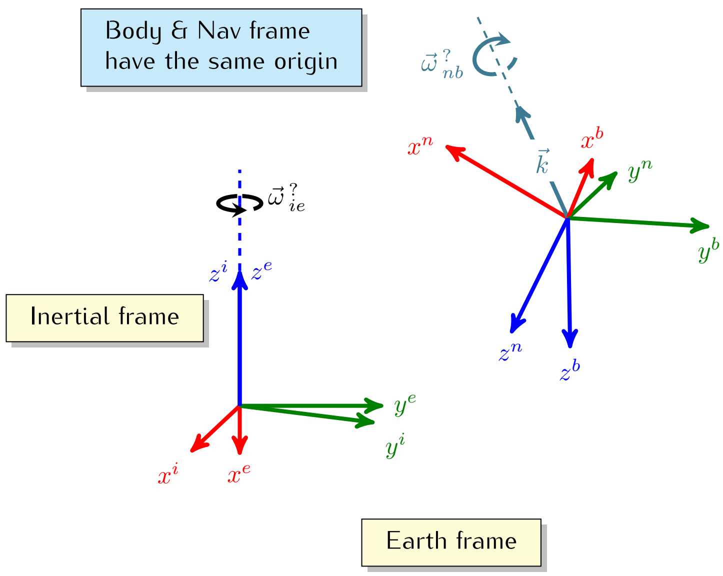
Body *wrt* Nav Frame



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Attitude — Method A

- Start with angular velocity

$$\vec{\omega}^{\,b}_{ib} = \vec{\omega}^{\,b}_{ie} + \vec{\omega}^{\,b}_{en} + \vec{\omega}^{\,b}_{nb} \rightarrow \vec{\omega}^{\,b}_{nb} = \vec{\omega}^{\,b}_{ib} - \vec{\omega}^{\,b}_{ie} - \vec{\omega}^{\,b}_{en}$$

- Now

$$\begin{aligned}\dot{C}^n_b &= C^n_b \Omega^b_{nb} = C^n_b \left(\Omega^b_{ib} - \Omega^b_{ie} - \Omega^b_{en}\right) \\ &= C^n_b \Omega^b_{ib} - C^n_b \Omega^b_{ie} - C^n_b \Omega^b_{en} \\ &= C^n_b \Omega^b_{ib} - \left(\Omega^n_{ie} + \Omega^n_{en}\right) C^n_b\end{aligned}$$

Recall that:
$[(C\vec{\omega})\times] = C[\vec{\omega}\times]C^T$
$C[\vec{\omega}\times] = [(C\vec{\omega})\times]C$

$$\omega^n_{ie} = C^n_e \omega^e_{ie} = \omega_{ie} \begin{bmatrix} \cos L_b \\ 0 \\ -\sin L_b \end{bmatrix}$$

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Attitude — Method A

- $\Omega^n_{en} = [\vec{\omega}^{\,n}_{en}\times]$
- Courtesy of Prof. Bruder and Mathematica

$$\dot{C}^e_n = C^e_n \Omega^n_{en} \rightarrow \Omega^n_{en} = \left(C^e_n\right)^T \dot{C}^e_n = \begin{bmatrix} 0 & \dot{\lambda}_b \sin L_b & -\dot{L}_b \\ -\dot{\lambda}_b \sin L_b & 0 & -\dot{\lambda}_b \cos L_b \\ \dot{L}_b & \dot{\lambda}_b \cos L_b & 0 \end{bmatrix}$$

- therefore,

$$\omega^n_{en} = \begin{bmatrix} \dot{\lambda}_b \cos L_b \\ -\dot{L}_b \\ -\dot{\lambda}_b \sin L_b \end{bmatrix}$$

- Finally since,

$$\begin{bmatrix} \dot{L}_b \\ \dot{\lambda}_b \\ \dot{h}_b \end{bmatrix} = \begin{bmatrix} \frac{\vec{v}^{\,n}_{eb,N}}{R_N + h_b} \\ \frac{\vec{v}^{\,n}_{eb,E}}{\cos(L_b)(R_E + h_b)} \\ -\vec{v}^{\,n}_{eb,D} \end{bmatrix}$$

- then

$$\omega^n_{en} = \begin{bmatrix} \frac{\vec{v}^{\,n}_{eb,E}}{R_E + h_b} \\ -\frac{\vec{v}^{\,n}_{eb,N}}{R_N + h_b} \\ -\frac{\tan(L_b)\vec{v}^{\,n}_{eb,E}}{R_E + h_b} \end{bmatrix} \qquad (1)$$

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Attitude — Method A

$$C_b^n(+) - C_b^n(-) \approx \Delta t \dot{C}_b^n$$

$$C_b^n(+) \approx C_b^n(-) + \Delta t \left[C_b^n \Omega_{ib}^b - \left(\Omega_{ie}^n + \Omega_{en}^n\right) C_b^n(-) \right]$$
$$= C_b^n(-) \left(\mathcal{I} + \Omega_{ib}^b \Delta t \right) - \left(\Omega_{ie}^n + \Omega_{en}^n\right) C_b^n(-) \Delta t$$

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Attitude — Method B

- Body orientation frame at time "k" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$
- Start with the angular velocity

$$\omega_{nb}^b = \omega_{ib}^b - \omega_{ie}^b - \omega_{en}^b$$

$$\Omega_{nb}^b = \Omega_{ib}^b - \Omega_{ie}^b - \Omega_{en}^b$$
$$= \Omega_{ib}^b - C_n^b \Omega_{ie}^n C_b^n - C_n^b \Omega_{en}^n C_b^n$$

$$C_b^n(+) = C_b^n(-) e^{\Omega_{nb}^b \Delta t}$$

$$C_b^n(+) = C_b^n(-) \left[\mathcal{I} + \sin(\Delta\theta) \mathfrak{K} + \left[1 - \cos(\Delta\theta)\right] \mathfrak{K}^2 \right]$$

$e^{\Omega_{nb}^b \Delta t} = e^{\mathfrak{K} \Delta\theta}$

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Attitude — Method C

- Body orientation frame at time "k" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$

$$\bar{q}_b^n(+) = \bar{q}_b^n(-) \otimes \Delta \bar{q}_{b(k)}^{b(k-1)}$$

$\vec{\omega}_{nb}^b \Delta t = \vec{k} \Delta\theta$

$$\Delta \bar{q}_{b(k)}^{b(k-1)} = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k} \sin(\frac{\Delta\theta}{2}) \end{bmatrix}$$

Need to periodically renormalize $\bar{q}$

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Attitude Update— Summary

- High fidelity

$$C_b^n(+) = C_b^n(-) \left[\mathcal{I} + \sin(\Delta\theta) \mathfrak{K} + \left[1 - \cos(\Delta\theta)\right] \mathfrak{K}^2 \right] \quad (2)$$

  or

$$\bar{q}_b^n(+) = \bar{q}_b^n(-) \otimes \Delta \bar{q}_{b(k)}^{b(k-1)}$$

$$\Delta \bar{q}_{b(k)}^{b(k-1)} = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k} \sin(\frac{\Delta\theta}{2}) \end{bmatrix} \quad (3)$$

- Low fidelity

$$C_b^n(+) \approx C_b^n(-) \left(\mathcal{I} + \Omega_{ib}^b \Delta t \right) - \left(\Omega_{ie}^n + \Omega_{en}^n\right) C_b^n(-) \Delta t \quad (4)$$

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Steps 2–4

2. Specific force transformation
   - Simply coordinatize the specific force

$$\vec{f}_{ib}^{\,n} = C_b^n(+) \vec{f}_{ib}^{\,b} \quad (5)$$

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Steps 2–4

3. Velocity update

$$\dot{\vec{v}}^{\,e}_{eb} = \vec{a}^{\,e}_{eb} = \vec{f}^{\,e}_{ib} + \vec{g}^{\,e}_{b} - 2\Omega^e_{ie}\vec{v}^{\,e}_{eb}$$

   - Note that: $\vec{v}^{\,n}_{eb} = C^n_e \vec{v}^{\,e}_{eb}$

$$\begin{aligned}
\dot{\vec{v}}^{\,n}_{eb} &= \dot{C}^n_e \vec{v}^{\,e}_{eb} + C^n_e \, \dot{\vec{v}}^{\,e}_{eb} \\
&= \Omega^n_{ne} C^n_e \vec{v}^{\,e}_{eb} + C^n_e \left(\vec{f}^{\,e}_{ib} + \vec{g}^{\,e}_{b} - 2\Omega^e_{ie}\vec{v}^{\,e}_{eb} \right) \\
&= \vec{f}^{\,n}_{ib} + \vec{g}^{\,n}_{b} - \Omega^n_{en}\vec{v}^{\,n}_{eb} - 2C^n_e \Omega^e_{ie}\vec{v}^{\,e}_{eb} \\
&= \vec{f}^{\,n}_{ib} + \vec{g}^{\,n}_{b} - \Omega^n_{en}\vec{v}^{\,n}_{eb} - 2\Omega^n_{ie} C^n_e \vec{v}^{\,e}_{eb} \\
&= \vec{f}^{\,n}_{ib} + \vec{g}^{\,n}_{b} - \left(\Omega^n_{en} + 2\Omega^n_{ie}\right)\vec{v}^{\,n}_{eb}
\end{aligned}$$

   - Finally,

$$\vec{v}^{\,n}_{eb}(+) = \vec{v}^{\,n}_{eb}(-) + \Delta t \left[\vec{f}^{\,n}_{ib} + \vec{g}^{\,n}_{b} - \left(\Omega^n_{en} + 2\Omega^n_{ie}\right)\vec{v}^{\,n}_{eb}(-) \right]$$

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Steps 2–4

4. Position update
   - Recalling that

$$\begin{bmatrix} \dot{L}_b \\ \dot{\lambda}_b \\ \dot{h}_b \end{bmatrix} = \begin{bmatrix} \frac{\vec{v}^{\,n}_{eb,N}}{R_N + h_b} \\ \frac{\vec{v}^{\,n}_{eb,E}}{\cos(L_b)(R_E + h_b)} \\ -\vec{v}^{\,n}_{eb,D} \end{bmatrix}$$

5. then

$$\begin{aligned}
h_b(+) &= h_b(-) - \Delta t \left[\vec{v}^{\,n}_{eb,D}\right] \\
L_b(+) &= L_b(-) + \Delta t \left[\frac{\vec{v}^{\,n}_{eb,N}}{R_N + h_b} \right] \\
\lambda_b(+) &= \lambda_b(-) + \Delta t \left[\frac{\vec{v}^{\,n}_{eb,E}}{\cos(L_b)(R_E + h_b)} \right]
\end{aligned}$$

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Nav Mechanization Summary

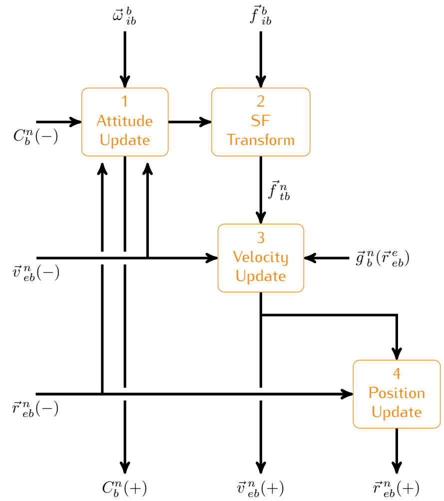
$$C_b^n(+) = C_b^n(-)\left[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1-\cos(\Delta\theta)]\,\mathfrak{K}^2\right] \tag{6}$$

or

$$\bar{q}\,^n_b(+) = \bar{q}\,^n_b(-) \otimes \Delta\bar{q}\,^{b(k-1)}_{b(k)}, \quad \Delta\bar{q}\,^{b(k-1)}_{b(k)} = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix} \tag{7}$$

or

$$C_b^n(+) \approx C_b^n(-)\left(\mathcal{I} + \Omega^b_{ib}\Delta t\right) - \left(\Omega^n_{ie} + \Omega^n_{en}\right) C_b^n(-)\Delta t \tag{8}$$



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Nav Mechanization Summary

$$\vec{f}\,^n_{ib} = C_b^n(+)\vec{f}\,^b_{ib} \tag{9}$$

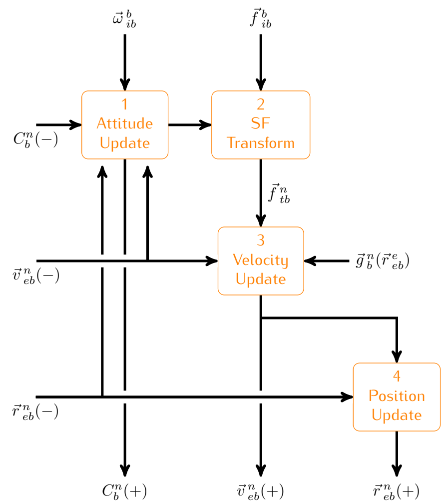
$$\vec{a}\,^n_{eb} = \vec{f}\,^n_{ib} + \vec{g}\,^n_b - (\Omega^n_{en} + 2\Omega^n_{ie})\vec{v}\,^n_{eb} \tag{10}$$

$$\vec{v}\,^n_{eb}(+) = \vec{v}\,^n_{eb}(-) + \Delta t\left[\vec{f}\,^n_{ib} + \vec{g}\,^n_b - (\Omega^n_{en} + 2\Omega^n_{ie})\,\vec{v}\,^n_{eb}(-)\right] \tag{11}$$

$$L_b(+) = L_b(-) + \Delta t\left[\frac{\vec{v}\,^n_{eb,N}}{R_N + h_b}\right] \tag{12}$$

$$\lambda_b(+) = \lambda_b(-) + \Delta t\left[\frac{\vec{v}\,^n_{eb,E}}{\cos(L_b)(R_E + h_b)}\right] \tag{13}$$

$$h_b(+) = h_b(-) - \Delta t\left[\vec{v}\,^n_{eb,D}\right] \tag{14}$$



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Nav Mechanization — Continuous Case

- In continuous time notation
  - Attitude: $\dot{C}_b^n = C_b^n\Omega^b_{ib} - (\Omega^n_{ie} + \Omega^n_{en})C_b^n$
  - Velocity: $\dot{\vec{v}}\,^n_{eb} = \vec{f}\,^n_{ib} + \vec{g}\,^n_b - (\Omega^n_{en} + 2\Omega^n_{ie})\vec{v}\,^n_{eb}$
  - Position:

$$\begin{bmatrix} \dot{L}_b \\ \dot{\lambda}_b \\ \dot{h}_b \end{bmatrix} = \begin{bmatrix} \frac{\vec{v}\,^n_{eb,N}}{R_N + h_b} \\ \frac{\vec{v}\,^n_{eb,E}}{\cos(L_b)(R_E + h_b)} \\ -\vec{v}\,^n_{eb,D} \end{bmatrix}$$

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Nav Mechanization — Continuous Case

- In State-space notation

$$\begin{bmatrix} \dot{L}_b \\ \dot{\lambda}_b \\ \dot{h}_b \\ \dot{\vec{v}}^{\,n}_{eb} \\ \dot{C}^n_b \end{bmatrix} = \begin{bmatrix} \frac{\vec{v}^{\,n}_{eb,N}}{R_N + h_b} \\ \frac{\vec{v}^{\,n}_{eb,E}}{\cos(L_b)(R_E + h_b)} \\ -\vec{v}^{\,n}_{eb,D} \\ \vec{f}^{\,n}_{ib} + \vec{g}^{\,n}_{b} - (\Omega^n_{en} + 2\Omega^n_{ie})\vec{v}^{\,n}_{eb} \\ C^n_b \Omega^b_{ib} - (\Omega^n_{ie} + \Omega^n_{en}) C^n_b \end{bmatrix} \tag{15}$$

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Appendix

$$[\bar{q}\otimes] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & -q_z & q_y \\ q_y & q_z & q_s & -q_x \\ q_z & -q_y & q_x & q_s \end{bmatrix}$$

$$[\bar{q}\circledast] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & q_z & -q_y \\ q_y & -q_z & q_s & q_x \\ q_z & q_y & -q_x & q_s \end{bmatrix}$$

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