

EE 565: Position, Navigation and Timing

Navigation Equations: Nav Mechanization

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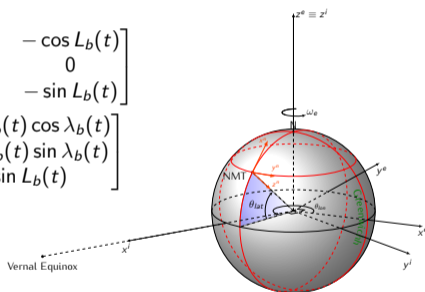
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- Determine the position, velocity and attitude of the **body** frame *wrt* the **Nav** frame.
 - **Position** — Typically described in curvilinear coordinates: $[L_b, \lambda_b, h_b]^T$
 - **Velocity** — Velocity of the body frame *wrt* the earth frame resolved in the navigation frame: \vec{v}_{eb}^n
 - **Attitude** — Orientation of the body frame *wrt* the navigation frame: C_b^n

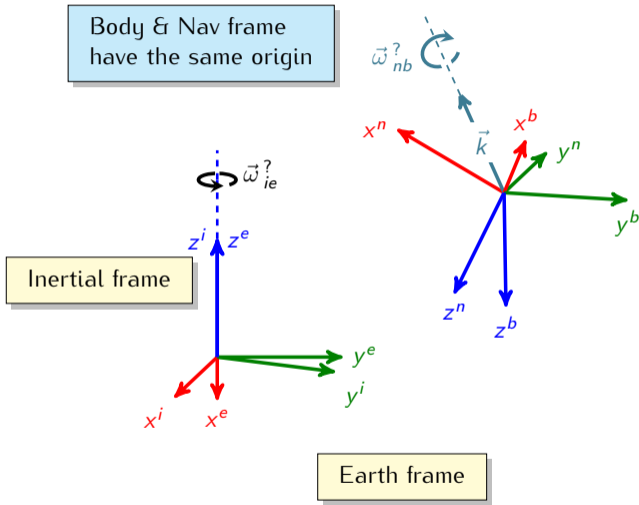
- Description of the Nav frame
 - Orientation of the n -frame wrt the e -frame

$$\begin{aligned}
 C_n^e(t) &= R_{(\vec{z}, \lambda_b(t))} R_{(\vec{y}, -L_b(t) - 90^\circ)} \\
 &= \begin{bmatrix} \cos \lambda_b(t) & -\sin \lambda_b(t) & 0 \\ \sin \lambda_b(t) & \cos \lambda_b(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin L_b(t) & 0 & -\cos L_b(t) \\ 0 & 1 & 0 \\ \cos L_b(t) & 0 & -\sin L_b(t) \end{bmatrix} \\
 &= \begin{bmatrix} -\sin L_b(t) \cos \lambda_b(t) & -\sin \lambda_b(t) & -\cos L_b(t) \cos \lambda_b(t) \\ -\sin L_b(t) \sin \lambda_b(t) & \cos \lambda_b(t) & -\cos L_b(t) \sin \lambda_b(t) \\ \cos L_b(t) & 0 & -\sin L_b(t) \end{bmatrix}
 \end{aligned}$$

where geodetic Lat = L_b and Geodetic Lon = λ_b



Body & Nav frame
have the same origin



- Start with angular velocity

$$\vec{\omega}_{ib}^b =$$

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- Now

$$\begin{aligned}\dot{C}_b^n &= C_b^n \Omega_{nb}^b = C_b^n \left(\Omega_{ib}^b - \Omega_{ie}^b - \Omega_{en}^b \right) \\ &= C_b^n \Omega_{ib}^b - C_b^n \Omega_{ie}^b - C_b^n \Omega_{en}^b \\ &= C_b^n \Omega_{ib}^b - (\Omega_{ie}^n + \Omega_{en}^n) C_b^n\end{aligned}$$

- Start with angular velocity

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Recall that:

$$\begin{aligned} [(C\vec{\omega}) \times] &= C[\vec{\omega} \times] C^T \\ C[\vec{\omega} \times] &= [(C\vec{\omega}) \times] C \end{aligned}$$

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$$\omega_{ie}^n = C_e^n \omega_{ie}^e = \omega_{ie} \begin{bmatrix} \cos L_b \\ 0 \\ -\sin L_b \end{bmatrix}$$

- $\Omega_{en}^n = [\vec{\omega}_{en}^n \times]$
- Courtesy of Prof. Bruder and Mathematica

$$\dot{C}_n^e = C_n^e \Omega_{en}^n \rightarrow \Omega_{en}^n = (C_n^e)^T \dot{C}_n^e = \begin{bmatrix} 0 & \dot{\lambda}_b \sin L_b & -\dot{L}_b \\ -\dot{\lambda}_b \sin L_b & 0 & -\dot{\lambda}_b \cos L_b \\ \dot{L}_b & \dot{\lambda}_b \cos L_b & 0 \end{bmatrix}$$

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- therefore,

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- therefore,

$$\omega_{en}^n = \begin{bmatrix} \dot{\lambda}_b \cos L_b \\ -\dot{L}_b \\ -\dot{\lambda}_b \sin L_b \end{bmatrix}$$

- Finally since,

$$\begin{bmatrix} \dot{L}_b \\ \dot{\lambda}_b \\ \dot{h}_b \end{bmatrix} = \begin{bmatrix} \frac{\vec{v}_{eb,N}^n}{R_N + h_b} \\ \frac{\vec{v}_{eb,E}^n}{\cos(L_b)(R_E + h_b)} \\ -\vec{v}_{eb,D}^n \end{bmatrix}$$

- then

$$\omega_{en}^n = \begin{bmatrix} \frac{\vec{v}_{eb,E}^n}{R_E + h_b} \\ -\frac{\vec{v}_{eb,N}^n}{R_N + h_b} \\ -\frac{\tan(L_b) \vec{v}_{eb,E}^n}{R_E + h_b} \end{bmatrix} \quad (1)$$

$$\begin{aligned}C_b^n(+)-C_b^n(-)&\approx \Delta t \dot{C}_b^n \\C_b^n(+)&\approx C_b^n(-)+\Delta t\left[C_b^n\Omega_{ib}^b-(\Omega_{ie}^n+\Omega_{en}^n)C_b^n(-)\right] \\&= C_b^n(-)\left(\mathcal{I}+\Omega_{ib}^b\Delta t\right)-\left(\Omega_{ie}^n+\Omega_{en}^n\right)C_b^n(-)\Delta t\end{aligned}$$

- Body orientation frame at time “ k ” wrt time “ $k - 1$ ”
 - $\Delta t = t_k - t_{k-1}$
- Start with the angular velocity

$$\omega_{nb}^b = \omega_{ib}^b - \omega_{ie}^b - \omega_{en}^b$$

$$\begin{aligned}\Omega_{nb}^b &= \Omega_{ib}^b - \Omega_{ie}^b - \Omega_{en}^b \\ &= \Omega_{ib}^b - C_n^b \Omega_{ie}^n C_b^n - C_n^b \Omega_{en}^n C_b^n\end{aligned}$$

$$C_b^n(+)=C_b^n(-)e^{\Omega_{nb}^b \Delta t}$$

$$C_b^n(+)=C_b^n(-)\left[\mathcal{I}+\sin(\Delta\theta)\mathfrak{K}+[1-\cos(\Delta\theta)]\mathfrak{K}^2\right]$$

$$e^{\Omega_{nb}^b \Delta t}=e^{\mathfrak{K}\Delta\theta}$$

- Body orientation frame at time “ k ” wrt time “ $k - 1$ ”
 - $\Delta t = t_k - t_{k-1}$

$$\bar{q}_b^n(+)=\bar{q}_b^n(-)\otimes\Delta\bar{q}_{b(k)}^{b(k-1)}$$

$$\vec{\omega}_{nb}^b\Delta t=\vec{k}\Delta\theta$$

$$\Delta\bar{q}_{b(k)}^{b(k-1)}=\begin{bmatrix}\cos\left(\frac{\Delta\theta}{2}\right) \\ \vec{k}\sin\left(\frac{\Delta\theta}{2}\right)\end{bmatrix}$$

Need to periodically renormalize \bar{q}

$$\vec{\omega}_{nb}^b \Delta t = \vec{k} \Delta \theta$$

$$\mathfrak{K} = [\vec{k} \times]$$

- High fidelity

$$C_b^n(+)=C_b^n(-)\left[\mathcal{I}+\sin(\Delta \theta) \mathfrak{K}+[1-\cos(\Delta \theta)] \mathfrak{K}^2\right] \quad (2)$$

or

$$\bar{q}_b^n(+)=\bar{q}_b^n(-) \otimes \Delta \bar{q}_{b(k)}^{b(k-1)}$$

$$\Delta \bar{q}_{b(k)}^{b(k-1)}=\left[\begin{array}{c} \cos\left(\frac{\Delta \theta}{2}\right) \\ \vec{k} \sin\left(\frac{\Delta \theta}{2}\right) \end{array}\right] \quad (3)$$

- Low fidelity

$$C_b^n(+)\approx C_b^n(-)\left(\mathcal{I}+\Omega_{ib}^b \Delta t\right)-\left(\Omega_{ie}^n+\Omega_{en}^n\right) C_b^n(-) \Delta t \quad (4)$$

- 2 Specific force transformation
 - Simply coordinatize the specific force

$$\vec{f}_{ib}^n = C_b^n(+)\vec{f}_{ib}^b \quad (5)$$

3 Velocity update

- Note that: $\vec{v}_{eb}^n = C_e^n \vec{v}_{eb}^e$

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3 Velocity update

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$$\dot{\vec{v}}_{eb}^e = \vec{a}_{eb}^e = \vec{f}_{ib}^e + \vec{g}_b^e - 2\Omega_{ie}^e \vec{v}_{eb}^e$$

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$$\begin{aligned} \dot{\vec{v}}_{eb}^n &= \dot{C}_e^n \vec{v}_{eb}^e + C_e^n \dot{\vec{v}}_{eb}^e \\ &= \Omega_{ne}^n C_e^n \vec{v}_{eb}^e + C_e^n \left(\vec{f}_{ib}^e + \vec{g}_b^e - 2\Omega_{ie}^e \vec{v}_{eb}^e \right) \end{aligned}$$

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$$\dot{\vec{v}}_{eb}^e = \vec{a}_{eb}^e = \vec{f}_{ib}^e + \vec{g}_b^e - 2\Omega_{ie}^e \vec{v}_{eb}^e$$

3 Velocity update

- Note that: $\vec{v}_{eb}^n = C_e^n \vec{v}_{eb}^e$

$$\begin{aligned} \dot{\vec{v}}_{eb}^n &= \dot{C}_e^n \vec{v}_{eb}^e + C_e^n \dot{\vec{v}}_{eb}^e \\ &= \Omega_{ne}^n C_e^n \vec{v}_{eb}^e + C_e^n \left(\vec{f}_{ib}^e + \vec{g}_b^e - 2\Omega_{ie}^e \vec{v}_{eb}^e \right) \\ &= \vec{f}_{ib}^n + \vec{g}_b^n - \Omega_{en}^n \vec{v}_{eb}^n - 2C_e^n \Omega_{ie}^e \vec{v}_{eb}^e \\ &= \vec{f}_{ib}^n + \vec{g}_b^n - \Omega_{en}^n \vec{v}_{eb}^n - 2\Omega_{ie}^n C_e^n \vec{v}_{eb}^e \end{aligned}$$

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$$\dot{\vec{v}}_{eb}^e = \vec{a}_{eb}^e = \vec{f}_{ib}^e + \vec{g}_b^e - 2\Omega_{ie}^e \vec{v}_{eb}^e$$

$$\begin{aligned}
 \dot{\vec{v}}_{eb}^n &= \dot{C}_e^n \vec{v}_{eb}^e + C_e^n \dot{\vec{v}}_{eb}^e \\
 &= \Omega_{ne}^n C_e^n \vec{v}_{eb}^e + C_e^n \left(\vec{f}_{ib}^e + \vec{g}_b^e - 2\Omega_{ie}^e \vec{v}_{eb}^e \right) \\
 &= \vec{f}_{ib}^n + \vec{g}_b^n - \Omega_{en}^n \vec{v}_{eb}^n - 2C_e^n \Omega_{ie}^e \vec{v}_{eb}^e \\
 &= \vec{f}_{ib}^n + \vec{g}_b^n - \Omega_{en}^n \vec{v}_{eb}^n - 2\Omega_{ie}^n C_e^n \vec{v}_{eb}^e \\
 &= \vec{f}_{ib}^n + \vec{g}_b^n - (\Omega_{en}^n + 2\Omega_{ie}^n) \vec{v}_{eb}^n
 \end{aligned}$$

$$\dot{\vec{v}}_{eb}^e = \vec{a}_{eb}^e = \vec{f}_{ib}^e + \vec{g}_b^e - 2\Omega_{ie}^e \vec{v}_{eb}^e$$

3 Velocity update

- Note that: $\vec{v}_{eb}^n = C_e^n \vec{v}_{eb}^e$

$$\begin{aligned} \dot{\vec{v}}_{eb}^n &= \dot{C}_e^n \vec{v}_{eb}^e + C_e^n \dot{\vec{v}}_{eb}^e \\ &= \Omega_{ne}^n C_e^n \vec{v}_{eb}^e + C_e^n \left(\vec{f}_{ib}^e + \vec{g}_b^e - 2\Omega_{ie}^e \vec{v}_{eb}^e \right) \\ &= \vec{f}_{ib}^n + \vec{g}_b^n - \Omega_{en}^n \vec{v}_{eb}^n - 2C_e^n \Omega_{ie}^e \vec{v}_{eb}^e \\ &= \vec{f}_{ib}^n + \vec{g}_b^n - \Omega_{en}^n \vec{v}_{eb}^n - 2\Omega_{ie}^n C_e^n \vec{v}_{eb}^e \\ &= \vec{f}_{ib}^n + \vec{g}_b^n - (\Omega_{en}^n + 2\Omega_{ie}^n) \vec{v}_{eb}^n \end{aligned}$$

- Finally,

$$\vec{v}_{eb}^n(+)=\vec{v}_{eb}^n(-)+\Delta t\left[\vec{f}_{ib}^n+\vec{g}_b^n-\left(\Omega_{en}^n+2\Omega_{ie}^n\right)\vec{v}_{eb}^n(-)\right]$$

- 4 Position update
- Recalling that

$$\begin{bmatrix} \dot{L}_b \\ \dot{\lambda}_b \\ \dot{h}_b \end{bmatrix} = \begin{bmatrix} \frac{\vec{v}_{eb,N}^n}{R_N + h_b} \\ \frac{\vec{v}_{eb,E}^n}{\cos(L_b)(R_E + h_b)} \\ -\vec{v}_{eb,D}^n \end{bmatrix}$$

- 5 then

$$h_b(+)=h_b(-)-\Delta t\left[\vec{v}_{eb,D}^n\right]$$

$$L_b(+)=L_b(-)+\Delta t\left[\frac{\vec{v}_{eb,N}^n}{R_N+h_b}\right]$$

$$\lambda_b(+)=\lambda_b(-)+\Delta t\left[\frac{\vec{v}_{eb,E}^n}{\cos(L_b)(R_E+h_b)}\right]$$

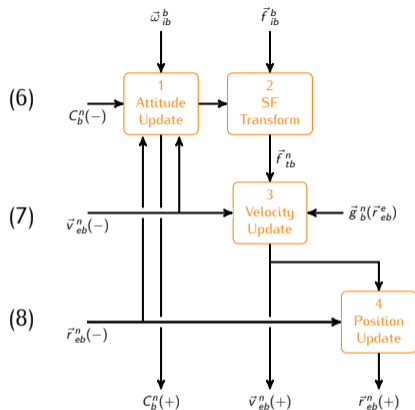
$$C_b^n(+)=C_b^n(-)\left[\mathcal{I}+\sin(\Delta\theta)\mathfrak{K}+[1-\cos(\Delta\theta)]\mathfrak{K}^2\right]$$

or

$$\bar{q}_b^n(+)=\bar{q}_b^n(-)\otimes\Delta\bar{q}_{b(k)}^{b(k-1)}, \quad \Delta\bar{q}_{b(k)}^{b(k-1)}=\begin{bmatrix} \cos\left(\frac{\Delta\theta}{2}\right) \\ \vec{k}\sin\left(\frac{\Delta\theta}{2}\right) \end{bmatrix}$$

or

$$C_b^n(+)\approx C_b^n(-)\left(\mathcal{I}+\Omega_{ib}^b\Delta t\right)-\left(\Omega_{ie}^n+\Omega_{en}^n\right)C_b^n(-)\Delta t$$



$$\vec{f}_{ib}^n = C_b^n(+)\vec{f}_{ib}^b \quad (9)$$

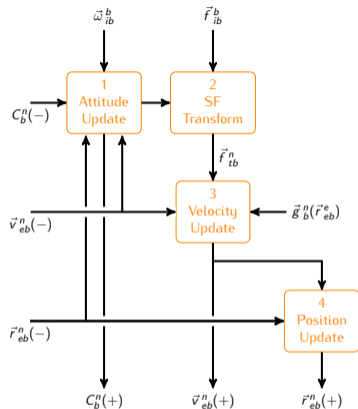
$$\vec{a}_{eb}^n = \vec{f}_{ib}^n + \vec{g}_b^n - (\Omega_{en}^n + 2\Omega_{ie}^n)\vec{v}_{eb}^n \quad (10)$$

$$\vec{v}_{eb}^n(+)=\vec{v}_{eb}^n(-)+\Delta t\left[\vec{f}_{ib}^n+\vec{g}_b^n-(\Omega_{en}^n+2\Omega_{ie}^n)\vec{v}_{eb}^n(-)\right] \quad (11)$$

$$L_b(+)=L_b(-)+\Delta t\left[\frac{\vec{v}_{eb,N}^n}{R_N+h_b}\right] \quad (12)$$

$$\lambda_b(+)=\lambda_b(-)+\Delta t\left[\frac{\vec{v}_{eb,E}^n}{\cos(L_b)(R_E+h_b)}\right] \quad (13)$$

$$h_b(+)=h_b(-)-\Delta t\left[\vec{v}_{eb,D}^n\right] \quad (14)$$



- In continuous time notation

- Attitude: $\dot{C}_b^n = C_b^n \Omega_{ib}^b - (\Omega_{ie}^n + \Omega_{en}^n) C_b^n$
- Velocity: $\dot{\vec{v}}_{eb}^n = \vec{f}_{ib}^n + \vec{g}_b^n - (\Omega_{en}^n + 2\Omega_{ie}^n) \vec{v}_{eb}^n$
- Position:

$$\begin{bmatrix} \dot{L}_b \\ \dot{\lambda}_b \\ \dot{h}_b \end{bmatrix} = \begin{bmatrix} \frac{\vec{v}_{eb,N}^n}{R_N + h_b} \\ \frac{\vec{v}_{eb,E}^n}{\cos(L_b)(R_E + h_b)} \\ -\vec{v}_{eb,D}^n \end{bmatrix}$$

- In State-space notation

$$\begin{bmatrix} \dot{L}_b \\ \dot{\lambda}_b \\ \dot{h}_b \\ \dot{\vec{v}}_{eb}^n \\ \dot{C}_b^n \end{bmatrix} = \begin{bmatrix} \frac{\vec{v}_{eb,N}^n}{R_N+h_b} \\ \frac{\vec{v}_{eb,E}^n}{\cos(L_b)(R_E+h_b)} \\ -\vec{v}_{eb,D}^n \\ \vec{f}_{ib}^n + \vec{g}_b^n - (\Omega_{en}^n + 2\Omega_{ie}^n)\vec{v}_{eb}^n \\ C_b^n\Omega_{ib}^b - (\Omega_{ie}^n + \Omega_{en}^n)C_b^n \end{bmatrix} \quad (15)$$

$$[\bar{q} \otimes] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & -q_z & q_y \\ q_y & q_z & q_s & -q_x \\ q_z & -q_y & q_x & q_s \end{bmatrix}$$

$$[\bar{q} \circledast] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & q_z & -q_y \\ q_y & -q_z & q_s & q_x \\ q_z & q_y & -q_x & q_s \end{bmatrix}$$