

# Lecture

## Navigation Equations: Tangential Mechanization

EE 565: Position, Navigation and Timing

Lecture Notes Update on Spring 2023

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### Tangential Mechanization

- Determine the position, velocity and attitude of the **body** frame *wrt* the **tangential** frame.
  - **Position** — Vector from the origin of the tangential frame to the origin of the body frame resolved in the tangential frame:  $\vec{r}_{tb}^t$
  - **Velocity** — Velocity of the body frame *wrt* the tangential frame resolved in the tangential frame:  $\vec{v}_{tb}^t$
  - **Attitude** — Orientation of the body frame *wrt* the tangential frame:  $C_b^t$

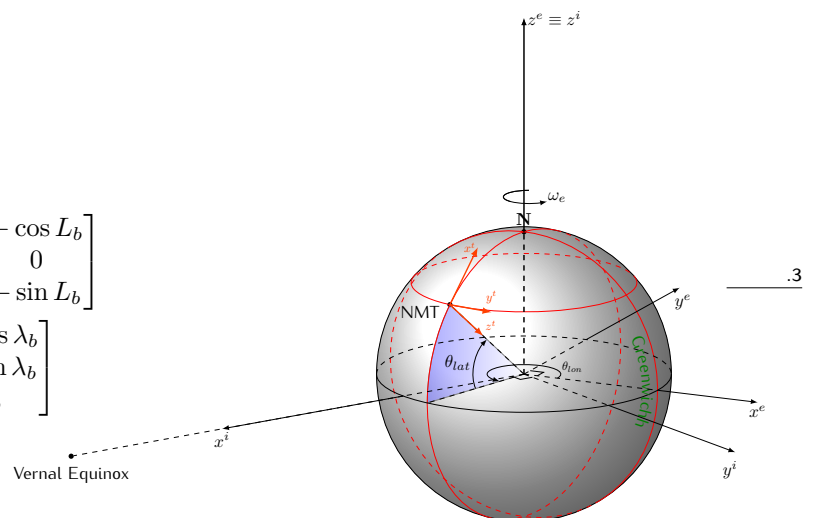
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### ECEF/Tangential

- Description of the tangential frame
  - Orientation of the *t*-frame *wrt* the *e*-frame

$$\begin{aligned}
 C_t^e &= R_{(z, \lambda_b)} R_{(y, -L_b - 90^\circ)} \\
 &= \begin{bmatrix} \cos \lambda_b & -\sin \lambda_b & 0 \\ \sin \lambda_b & \cos \lambda_b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin L_b & 0 & -\cos L_b \\ 0 & 1 & 0 \\ \cos L_b & 0 & -\sin L_b \end{bmatrix} \\
 &= \begin{bmatrix} -\sin L_b \cos \lambda_b & -\sin \lambda_b & -\cos L_b \cos \lambda_b \\ -\sin L_b \sin \lambda_b & \cos \lambda_b & -\cos L_b \sin \lambda_b \\ \cos L_b & 0 & -\sin L_b \end{bmatrix}
 \end{aligned}$$

where geodetic Lat =  $L_b$  and Geodetic Lon =  $\lambda_b$



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### Attitude — Method A

- Body orientation frame at time “*k*” *wrt* time “*k* – 1”
  - $\Delta t = t_k - t_{k-1}$

- Start with angular velocity

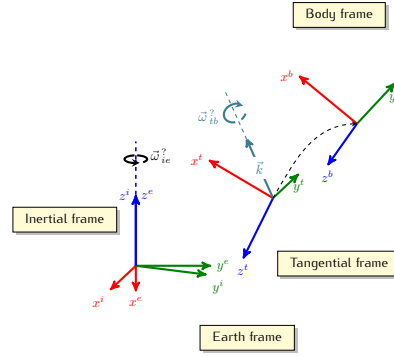
$$\vec{\omega}_{ib}^t = \vec{\omega}_{ie}^t + \vec{\omega}_{et}^t + \vec{\omega}_{tb}^t$$

$$\vec{\omega}_{tb}^t = C_b^t \vec{\omega}_{ib}^b - C_e^t \vec{\omega}_{ie}^e$$

$$\Omega_{tb}^t = C_b^t \Omega_{ib}^b C_t^b - C_e^t \Omega_{ie}^e C_t^e$$

$$C_b^t(+)-C_b^t(-) \approx \Delta t \Omega_{tb}^t C_b^t(-)$$

$$\begin{aligned} C_b^t(+)&\approx C_b^t(-) + \Delta t (C_b^t \Omega_{ib}^b C_t^b - C_e^t \Omega_{ie}^e C_t^e) C_b^t(-) \\ &= C_b^t(-) (\mathcal{I} + \Omega_{ib}^b \Delta t) - \Omega_{ie}^e C_b^t(-) \Delta t \end{aligned}$$



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### Attitude — Method B

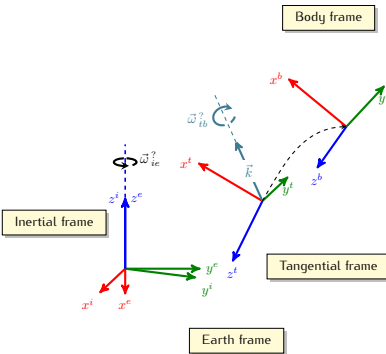
- Body orientation frame at time “k” wrt time “k - 1”  
-  $\Delta t = t_k - t_{k-1}$
- Start with the angular velocity

$$\Omega_{tb}^t = C_b^t \Omega_{ib}^b C_t^b - \Omega_{ie}^t$$

$$C_b^t(+)=C_b^t(-)e^{\Omega_{ib}^b \Delta t}=e^{\Omega_{tb}^t \Delta t} C_b^t(-)$$

$$C_b^t(+)=[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1 - \cos(\Delta\theta)]\mathfrak{K}^2] C_b^t(-)$$

$$e^{\Omega_{tb}^t \Delta t} = e^{\mathfrak{K}\theta}$$



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### Attitude — Method C

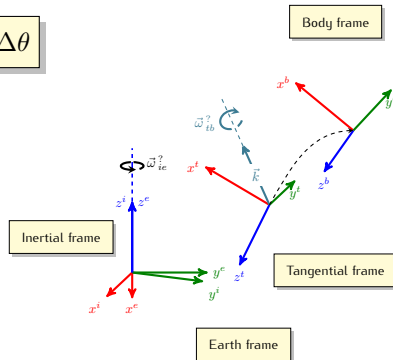
- Body orientation frame at time “k” wrt time “k - 1”  
-  $\Delta t = t_k - t_{k-1}$

$$\vec{\omega}_{tb}^t \Delta t = \vec{k} \Delta\theta$$

$$\bar{q}_b^t(+)=\Delta\bar{q}_b^t \otimes \bar{q}_b^t(-)$$

$$\Delta\bar{q}_b^t = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k} \sin(\frac{\Delta\theta}{2}) \end{bmatrix}$$

Need to periodically renormalize  $\bar{q}$



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### Attitude Update— Summary

- High fidelity

$$C_b^t(+)= [\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1 - \cos(\Delta\theta)]\mathfrak{K}^2] C_b^t(-) \quad (1)$$

or

$$\bar{q}_b^t(+)= \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k} \sin(\frac{\Delta\theta}{2}) \end{bmatrix} \otimes \bar{q}_b^t(-) \quad (2)$$

- Low fidelity

$$C_b^t(+)\approx C_b^t(-) (\mathcal{I} + \Omega_{ib}^b \Delta t) - \Omega_{ie}^e C_b^t(-) \Delta t \quad (3)$$

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Steps 2-4

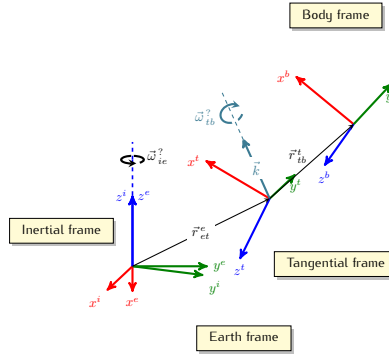
2. Specific force transformation

- Simply coordinatize the specific force

$$\vec{f}_{ib}^t = C_b^t(+)\vec{f}_{ib}^b \quad (4)$$

3. Velocity update

$$\begin{aligned} \vec{r}_{ib}^i &= \cancel{\vec{r}_{ie}^i} + C_e^i \vec{r}_{et}^e + C_t^i \vec{r}_{tb}^t \\ \Rightarrow \vec{r}_{tb}^t &= C_i^t \vec{r}_{ib}^i - C_e^t \vec{r}_{et}^e \\ \vec{v}_{tb}^t &= \dot{\vec{r}}_{tb}^t \\ &= \dot{C}_i^t \vec{r}_{ib}^i + C_i^t \dot{\vec{r}}_{ib}^i \\ &= \Omega_{ei}^t C_i^t \vec{r}_{ib}^i + C_i^t \vec{v}_{ib}^i \\ &= -\Omega_{ie}^t (\vec{r}_{et}^t + \vec{r}_{tb}^t) + C_i^t \vec{v}_{ib}^i \end{aligned}$$



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Steps 2-4

$$\begin{aligned} \vec{a}_{tb}^t &= \dot{\vec{v}}_{tb}^t = \frac{d}{dt} (-\Omega_{ie}^t (\vec{r}_{et}^t + \vec{r}_{tb}^t) + C_i^t \vec{v}_{ib}^i) \\ &= -\Omega_{ie}^t \dot{\vec{r}}_{tb}^t + \dot{C}_i^t \vec{v}_{ib}^i + C_i^t \dot{\vec{v}}_{ib}^i \\ &= -\Omega_{ie}^t \vec{v}_{tb}^t + \Omega_{ti}^t C_i^t \vec{v}_{ib}^i + C_i^t \vec{a}_{ib}^i \\ &= -\Omega_{ie}^t \vec{v}_{tb}^t - \Omega_{it}^t [\vec{v}_{tb}^t + \Omega_{ie}^t (\vec{r}_{et}^t + \vec{r}_{tb}^t)] + C_i^t \vec{a}_{ib}^i \\ &= -2\Omega_{ie}^t \vec{v}_{tb}^t - \Omega_{ie}^t \Omega_{ie}^t \vec{r}_{eb}^t + \vec{a}_{ib}^t \\ &= -2\Omega_{ie}^t \vec{v}_{tb}^t + \vec{f}_{ib}^t + \vec{g}_b^t \end{aligned}$$

$\dot{\vec{r}}_{et}^t = 0$

$\vec{\omega}_{it}^t = \vec{\omega}_{ie}^t$

$C_i^t \vec{v}_{ib}^i = \Omega_{ie}^t (\vec{r}_{et}^t + \vec{r}_{tb}^t) + \vec{v}_{tb}^t$

$\vec{f}_{ib}^t = \vec{a}_{ib}^t - \vec{\gamma}_{ib}^t$

$\vec{a}_{ib}^t = \vec{f}_{ib}^t + \vec{\gamma}_{ib}^t$

$\vec{g}_b^t = \vec{\gamma}_{ib}^t - \Omega_{ie}^t \Omega_{ie}^t \vec{r}_{eb}^t$

$\vec{a}_{ib}^t = \vec{f}_{ib}^t + \vec{g}_b^t + \Omega_{ie}^t \Omega_{ie}^t \vec{r}_{eb}^t$

$$\begin{aligned} \vec{v}_{tb}^t(+)&= \vec{v}_{tb}^t(-) + \vec{a}_{tb}^t \Delta t \\ &= \vec{v}_{tb}^t(-) + \left[ \vec{f}_{ib}^t + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^t(-) \right] \Delta t \end{aligned} \quad (5)$$

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Steps 2-4

4. Position update

- by simple numerical integration

$$\vec{r}_{tb}^t(+)=\vec{r}_{tb}^t(-)+\vec{v}_{tb}^t(-)\Delta t+\vec{a}_{tb}^t\frac{\Delta t^2}{2} \quad (6)$$

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## Tangential Mechanization Summary

$$C_b^t(+)= [\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1 - \cos(\Delta\theta)]\mathfrak{K}^2] C_b^t(-)$$

or

$$\bar{q}_b^t(+)= \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k} \sin(\frac{\Delta\theta}{2}) \end{bmatrix} \otimes \bar{q}_b^t(-)$$

or

$$C_b^t(+)\approx C_b^t(-) (\mathcal{I} + \Omega_{ib}^b \Delta t) - \Omega_{ie}^t C_b^t(-) \Delta t$$

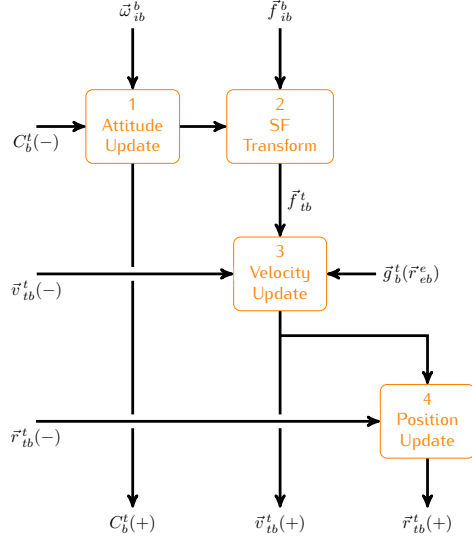
and

$$\vec{f}_{ib}^t = C_b^t(+)\vec{f}_{ib}^b$$

$$\vec{a}_{tb}^t = -2\Omega_{ie}^t \vec{v}_{tb}^t + \vec{f}_{ib}^t + \vec{g}_b^t$$

$$\vec{v}_{tb}^t(+)= \vec{v}_{tb}^t(-) + [\vec{f}_{ib}^t + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^t(-)] \Delta t$$

$$\vec{r}_{tb}^t(+)= \vec{r}_{tb}^t(-) + \vec{v}_{tb}^t(-)\Delta t + \vec{a}_{tb}^t \frac{\Delta t^2}{2}$$



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## Tangential Mechanization — Continuous Case

- In continuous time notation

- Attitude:  $\dot{C}_b^t = C_b^t \Omega_{tb}^b$  or  $\dot{\bar{q}}_b^t = \frac{1}{2}[\tilde{\omega}_{tb}^b \otimes] \bar{q}_b^t(t)$

- Velocity:  $\dot{\vec{v}}_{tb}^t = C_b^t \vec{f}_{ib}^b + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^t$

- Position:  $\dot{\vec{r}}_{tb}^t = \vec{v}_{tb}^t$

- In State-space notation

$$\begin{bmatrix} \dot{\vec{r}}_{tb}^t \\ \dot{\vec{v}}_{tb}^t \\ \dot{C}_b^t \end{bmatrix} = \begin{bmatrix} \vec{v}_{tb}^t \\ C_b^t \vec{f}_{ib}^b + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^t \\ C_b^t \Omega_{tb}^b \end{bmatrix} \quad (7)$$

or

$$\begin{bmatrix} \dot{\vec{r}}_{tb}^t \\ \dot{\vec{v}}_{tb}^t \\ \dot{\bar{q}}_b^t \end{bmatrix} = \begin{bmatrix} \vec{v}_{tb}^t \\ C_b^t \vec{f}_{ib}^b + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^t \\ \frac{1}{2}[\tilde{\omega}_{tb}^b \otimes] \bar{q}_b^t(t) \end{bmatrix} \quad (8)$$

where  $\tilde{\omega}_{ib}^b = \tilde{\omega}_{ie}^b + \tilde{\omega}_{eb}^b$ , and  $\Omega_{eb}^b = \Omega_{ib}^b - \Omega_{ie}^b$ .

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## Appendix

$$[\bar{q} \otimes] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & -q_z & q_y \\ q_y & q_z & q_s & -q_x \\ q_z & -q_y & q_x & q_s \end{bmatrix}$$

$$[\bar{q} \otimes] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & q_z & -q_y \\ q_y & -q_z & q_s & q_x \\ q_z & q_y & -q_x & q_s \end{bmatrix}$$

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