

EE 565: Position, Navigation and Timing

Navigation Equations: Tangential Mechanization

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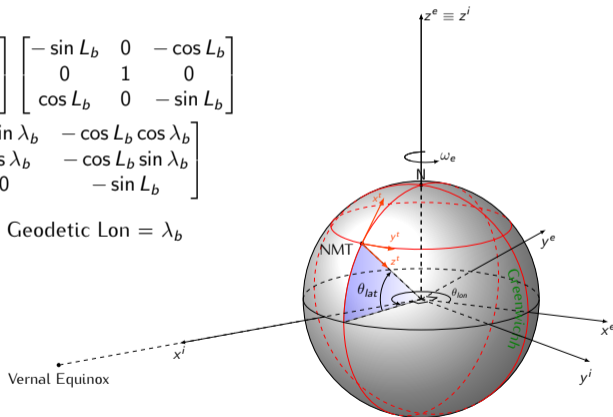
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- Determine the position, velocity and attitude of the **body** frame *wrt* the **tangential** frame.
 - **Position** — Vector from the origin of the tangential frame to the origin of the body frame resolved in the tangential frame: \vec{r}_{tb}^t
 - **Velocity** — Velocity of the body frame *wrt* the tangential frame resolved in the tangential frame: \vec{v}_{tb}^t
 - **Attitude** — Orientation of the body frame *wrt* the tangential frame: C_b^t

- Description of the tangential frame
 - Orientation of the t -frame wrt the e -frame

$$\begin{aligned}
 C_t^e &= R_{(z, \lambda_b)} R_{(y, -L_b - 90^\circ)} \\
 &= \begin{bmatrix} \cos \lambda_b & -\sin \lambda_b & 0 \\ \sin \lambda_b & \cos \lambda_b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin L_b & 0 & -\cos L_b \\ 0 & 1 & 0 \\ \cos L_b & 0 & -\sin L_b \end{bmatrix} \\
 &= \begin{bmatrix} -\sin L_b \cos \lambda_b & -\sin \lambda_b & -\cos L_b \cos \lambda_b \\ -\sin L_b \sin \lambda_b & \cos \lambda_b & -\cos L_b \sin \lambda_b \\ \cos L_b & 0 & -\sin L_b \end{bmatrix}
 \end{aligned}$$

where geodetic Lat = L_b and Geodetic Lon = λ_b



- Body orientation frame at time “ k ” wrt time “ $k - 1$ ”
 - $\Delta t = t_k - t_{k-1}$
- Start with angular velocity

$$\vec{\omega}_{ib}^t = \vec{\omega}_{ie}^t + \vec{\omega}_{et}^t + \vec{\omega}_{tb}^t$$

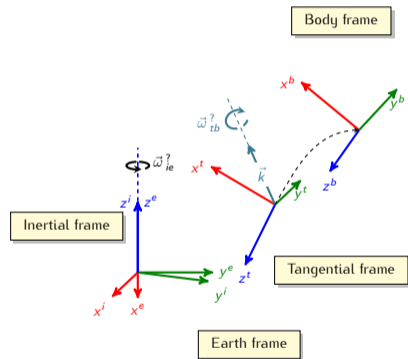
$$\vec{\omega}_{tb}^t = C_b^t \vec{\omega}_{ib}^b - C_e^t \vec{\omega}_{ie}^e$$

$$\Omega_{tb}^t = C_b^t \Omega_{ib}^b C_t^b - C_e^t \Omega_{ie}^e C_t^e$$

$$C_b^t(+)- C_b^t(-) \approx \Delta t \Omega_{tb}^t C_b^t(-)$$

$$C_b^t(+)- C_b^t(-) \approx C_b^t(-) + \Delta t \left(C_b^t \Omega_{ib}^b C_t^b - C_e^t \Omega_{ie}^e C_t^e \right) C_b^t(-)$$

$$= C_b^t(-) \left(\mathcal{I} + \Omega_{ib}^b \Delta t \right) - \Omega_{ie}^e C_b^t(-) \Delta t$$



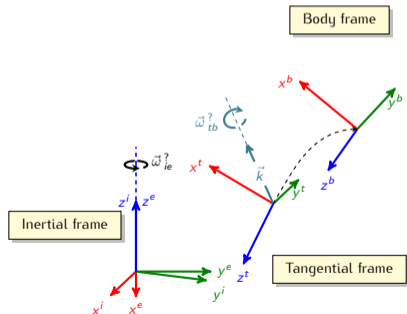
- Body orientation frame at time “ k ” wrt time “ $k - 1$ ”
 - $\Delta t = t_k - t_{k-1}$
- Start with the angular velocity

$$\Omega_{tb}^t = C_b^t \Omega_{ib}^b C_t^b - \Omega_{ie}^t$$

$$C_b^t(+)=C_b^t(-)e^{\Omega_{tb}^b \Delta t}=e^{\Omega_{tb}^t \Delta t} C_b^t(-)$$

$$C_b^t(+)=\left[\mathcal{I}+\sin(\Delta\theta)\hat{\mathcal{K}}+[1-\cos(\Delta\theta)]\hat{\mathcal{K}}^2\right]C_b^t(-)$$

$$e^{\Omega_{tb}^t \Delta t}=e^{\hat{\mathcal{K}}\theta}$$



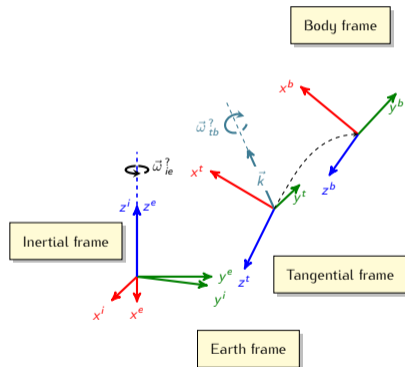
- Body orientation frame at time “ k ” wrt time “ $k - 1$ ”
 - $\Delta t = t_k - t_{k-1}$

$$\vec{\omega}_{tb}^t \Delta t = \vec{k} \Delta \theta$$

$$\bar{q}_b^t(+)=\Delta \bar{q}_b^t \otimes \bar{q}_b^t(-)$$

$$\Delta \bar{q}_b^t = \begin{bmatrix} \cos\left(\frac{\Delta \theta}{2}\right) \\ \vec{k} \sin\left(\frac{\Delta \theta}{2}\right) \end{bmatrix}$$

Need to periodically renormalize \bar{q}



$$\vec{\omega}_{tb}^t \Delta t = \vec{k} \Delta \theta$$

$$\mathfrak{K} = [\vec{k} \times]$$

- High fidelity

$$C_b^t(+)= [\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1 - \cos(\Delta\theta)] \mathfrak{K}^2] C_b^t(-) \quad (1)$$

or

$$\bar{q}_b^t(+)= \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k} \sin(\frac{\Delta\theta}{2}) \end{bmatrix} \otimes \bar{q}_b^t(-) \quad (2)$$

- Low fidelity

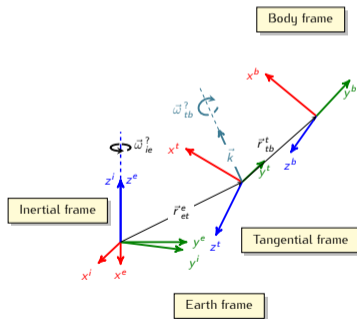
$$C_b^t(+)\approx C_b^t(-) (\mathcal{I} + \Omega_{ib}^b \Delta t) - \Omega_{ie}^t C_b^t(-) \Delta t \quad (3)$$

- 2 Specific force transformation
 - Simply coordinatize the specific force

$$\vec{f}_{ib}^t = C_b^t(+)\vec{f}_{ib}^b \quad (4)$$

- 3 Velocity update

$$\begin{aligned} \vec{r}_{ib}^i &= \cancel{\vec{r}_{ie}^i} + C_e^i \vec{r}_{et}^e + C_t^i \vec{r}_{tb}^t \\ \Rightarrow \vec{r}_{tb}^t &= C_i^t \vec{r}_{ib}^i - C_e^t \vec{r}_{et}^e \\ \vec{v}_{tb}^t &= \dot{\vec{r}}_{tb}^t \\ &= \dot{C}_i^t \vec{r}_{ib}^i + C_i^t \dot{\vec{r}}_{ib}^i \\ &= \Omega_{ei}^t C_i^t \vec{r}_{ib}^i + C_i^t \vec{v}_{ib}^i \\ &= -\Omega_{ie}^t (\vec{r}_{et}^t + \vec{r}_{tb}^t) + C_i^t \vec{v}_{ib}^i \end{aligned}$$



$$\vec{a}_{tb}^t = \dot{\vec{v}}_{tb}^t = \frac{d}{dt} (-\Omega_{ie}^t (\vec{r}_{et}^t + \vec{r}_{tb}^t) + C_i^t \vec{v}_{ib}^i)$$

$$= -\Omega_{ie}^t \dot{\vec{r}}_{tb}^t + \dot{C}_i^t \vec{v}_{ib}^i + C_i^t \dot{\vec{v}}_{ib}^i$$

$$= -\Omega_{ie}^t \vec{v}_{tb}^t + \Omega_{ti}^t C_i^t \vec{v}_{ib}^i + C_i^t \vec{a}_{ib}^i$$

$$= -\Omega_{ie}^t \vec{v}_{tb}^t - \Omega_{it}^t [\vec{v}_{tb}^t + \Omega_{ie}^t (\vec{r}_{et}^t + \vec{r}_{tb}^t)] + C_i^t \vec{a}_{ib}^i$$

$$= -2\Omega_{ie}^t \vec{v}_{tb}^t - \Omega_{ie}^t \Omega_{ie}^t \vec{r}_{eb}^t + \vec{a}_{ib}^t$$

$$= -2\Omega_{ie}^t \vec{v}_{tb}^t + \vec{f}_{ib}^t + \vec{g}_b^t$$

$$\vec{v}_{tb}^t(+) = \vec{v}_{tb}^t(-) + \vec{a}_{tb}^t \Delta t$$

$$= \vec{v}_{tb}^t(-) + \left[\vec{f}_{ib}^t + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^t(-) \right] \Delta t \quad (5)$$

$$\dot{\vec{r}}_{et}^t = 0$$

$$\vec{\omega}_{it}^t = \vec{\omega}_{ie}^t$$

$$C_i^t \vec{v}_{ib}^i = \Omega_{ie}^t (\vec{r}_{et}^t + \vec{r}_{tb}^t) + \vec{v}_{tb}^t$$

$$\vec{f}_{ib}^? = \vec{a}_{ib}^? - \vec{\gamma}_{ib}^?$$

$$\vec{a}_{ib}^t = \vec{f}_{ib}^t + \vec{\gamma}_{ib}^t$$

$$\vec{g}_b^t = \vec{\gamma}_{ib}^t - \Omega_{ie}^t \Omega_{ie}^t \vec{r}_{eb}^t$$

$$\vec{a}_{ib}^t = \vec{f}_{ib}^t + \vec{g}_b^t + \Omega_{ie}^t \Omega_{ie}^t \vec{r}_{eb}^t$$

- ④ Position update
 - by simple numerical integration

$$\vec{r}_{tb}^t(+)=\vec{r}_{tb}^t(-)+\vec{v}_{tb}^t(-)\Delta t+\vec{a}_{tb}^t\frac{\Delta t^2}{2}\quad (6)$$

$$C_b^t(+)= [\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1 - \cos(\Delta\theta)] \mathfrak{K}^2] C_b^t(-)$$

or

$$\bar{q}_b^t(+)= \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k} \sin(\frac{\Delta\theta}{2}) \end{bmatrix} \otimes \bar{q}_b^t(-)$$

or

$$C_b^t(+)\approx C_b^t(-) (\mathcal{I} + \Omega_{ib}^b \Delta t) - \Omega_{ie}^t C_b^t(-) \Delta t$$

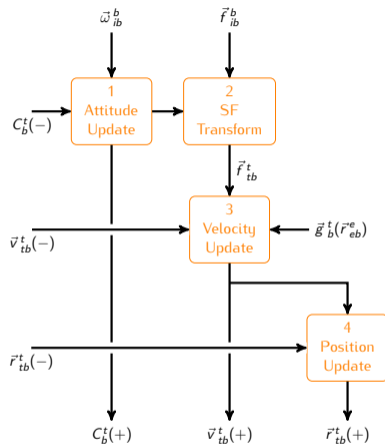
and

$$\vec{f}_{ib}^t = C_b^t(+)\vec{f}_{ib}^b$$

$$\vec{a}_{tb}^t = -2\Omega_{ie}^t \vec{v}_{tb}^t + \vec{f}_{ib}^t + \vec{g}_b^t$$

$$\vec{v}_{tb}^t(+)= \vec{v}_{tb}^t(-) + [\vec{f}_{ib}^t + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^t(-)] \Delta t$$

$$\vec{r}_{tb}^t(+)= \vec{r}_{tb}^t(-) + \vec{v}_{tb}^t(-)\Delta t + \vec{a}_{tb}^t \frac{\Delta t^2}{2}$$



- In continuous time notation

- Attitude: $\dot{C}_b^t = C_b^t \Omega_{tb}^b$ or $\dot{\bar{q}}_b^t = \frac{1}{2}[\tilde{\omega}_{tb}^b \otimes] \bar{q}_b^t(t)$

- Velocity: $\dot{\vec{v}}_{tb}^t = C_b^t \vec{f}_{ib}^b + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^t$

- Position: $\dot{\vec{r}}_{ib}^t = \vec{v}_{tb}^t$

$$\vec{\omega}_{ib}^b = \vec{\omega}_{ie}^b + \vec{\omega}_{tb}^b$$

$$\Omega_{tb}^b = \Omega_{ib}^b - \Omega_{ie}^b$$

- In State-space notation

$$\begin{bmatrix} \dot{\vec{r}}_{tb}^t \\ \dot{\vec{v}}_{tb}^t \\ \dot{C}_b^t \end{bmatrix} = \begin{bmatrix} \vec{v}_{tb}^t \\ C_b^t \vec{f}_{ib}^b + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^t \\ C_b^t \Omega_{tb}^b \end{bmatrix} \quad (7)$$

or

$$\begin{bmatrix} \dot{\vec{r}}_{tb}^t \\ \dot{\vec{v}}_{tb}^t \\ \dot{\bar{q}}_b^t \end{bmatrix} = \begin{bmatrix} \vec{v}_{tb}^t \\ C_b^t \vec{f}_{ib}^b + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^t \\ \frac{1}{2}[\tilde{\omega}_{tb}^b \otimes] \bar{q}_b^t(t) \end{bmatrix} \quad (8)$$

$$[\bar{q} \otimes] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & -q_z & q_y \\ q_y & q_z & q_s & -q_x \\ q_z & -q_y & q_x & q_s \end{bmatrix}$$

$$[\bar{q} \circledast] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & q_z & -q_y \\ q_y & -q_z & q_s & q_x \\ q_z & q_y & -q_x & q_s \end{bmatrix}$$