

Lecture

Power Spectral Density Estimation

EE 565: Position, Navigation, and Timing

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Motivation

Sensors suffer from noise effects that can not be removed through calibration, consequently, we need to

- understand the nature of the noise
- be able to extract parameters from actual data
- develop models to mimic noise in simulation to provide performance capabilities

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Purpose

Estimate the distribution of power in a signal. Unfortunately, truth and what is practical cause a problem.

Truth

- Infinitely long.
- Continuous in time and value.
- Provides true distribution of power.

Practice

- Finite length.
- Discrete in time and value.
- Only approximation of distribution of power.

Let's make it more interesting

The signal is stochastic in nature.

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1 Review Material

1.1 Signal Classification

Energy and Power

Assume the voltage across a resistor R is $e(t)$ and is producing a current $i(t)$. The instantaneous power per ohm is $p(t) = e(t)i(t)/R = i^2(t)$.

Total Energy

$$E = \lim_{T \rightarrow \infty} \int_{-T}^T i^2(t) dt \quad (1)$$

Average Power

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T i^2(t) dt \quad (2)$$

Arbitrary signal $x(t)$

Total Normalized Energy

$$E \triangleq \lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad (3)$$

Normalized Power

$$P \triangleq \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt \quad (4)$$

- $x(t)$ is an *energy signal* iff $0 < E < \infty$, so that $P = 0$.
- $x(t)$ is a *power signal* iff $0 < P < \infty$, so that $E = \infty$.

1.2 Time Averages

Correlation

For Energy Signals

$$\phi(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau) dt \quad (5)$$

Provides a measure of similarity or coherence between a signal and a delayed version of itself. Note that $\phi(0) = E$

For Power Signals

$$R(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t)x(t+\tau) dt \quad (6)$$

For Periodic Signals

$$R(\tau) = \frac{1}{T_0} \int_{T_0} x(t)x(t+\tau) dt \quad (7)$$

1.3 Frequency Domain

Energy Spectral Density

Rayleigh's Energy Theorem or Parseval's theorem

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(F)|^2 dF \quad (8)$$

Energy Spectral Density

$$G(F) \triangleq |X(F)|^2 \quad (9)$$

with units of $\text{volts}^2\text{-sec}^2$ or, if considered on a per-ohm basis, $\text{watts-sec/Hz}=\text{joules/Hz}$

Power Spectral Density

$$P = \int_{-\infty}^{\infty} S(F)dF = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt \quad (10)$$

where we define $S(F)$ as the power spectral density with units of watts/Hz. Note that $R(0) = \int_{-\infty}^{\infty} S(F)dF$.

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2 Random Signals and Noise

Basic Definitions

- Define an *experiment* with random *outcome*.
- Mapping of the outcome to a variable \Rightarrow random variable.
- Mapping of the outcome to a function \Rightarrow random function.

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2.1 Statistical Averages

Probability (Cumulative) Distribution Function (cdf)

$$F_X(x) = \text{probability that } X \leq x = P(X \leq x) \quad (11)$$

Describes the manner random variables take different values.

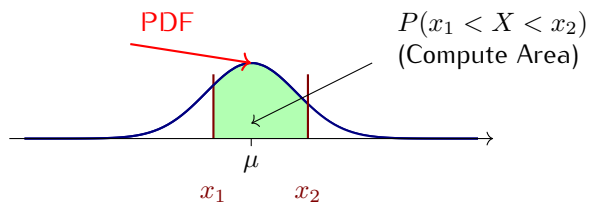
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Probability Density Function (pdf)

$$f_X(x) = \frac{dF_X(x)}{dx} \quad (12)$$

and

$$P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)dx \quad (13)$$



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PDF of Discrete Random Variables

If the random variable X takes a set of discrete values x_i with probability p_i , the pdf of X is expressed in terms of Dirac delta functions, i.e.,

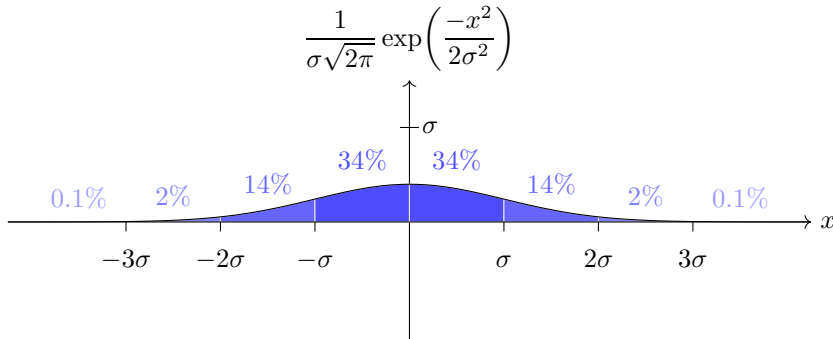
$$f_X(x) = \sum_i p_i \delta(x - x_i) \quad (14)$$

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Gaussian Distribution

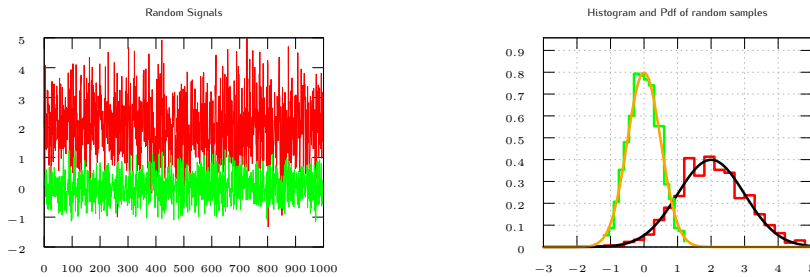
$$f_X(x) = \frac{1}{\sigma_x \sqrt{2\pi}} \exp \left[-\frac{(x - \mu_x)^2}{2\sigma_x^2} \right] \quad (15)$$

For example if $\sigma_x = \sigma$ and $\mu_x = 0$



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PDF of White Noise



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Mean and Variance

Mean of a Discrete RV

$$\bar{X} = \mathbb{E}[X] = \sum_{j=1}^M x_j P_j \quad (16)$$

Mean of a Continuous RV

$$\bar{X} = \mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx \quad (17)$$

Variance of a RV

$$\sigma_X^2 \triangleq \mathbb{E} \{ [X - \mathbb{E}(X)]^2 \} = \mathbb{E}[X^2] - \mathbb{E}^2[X] \quad (18)$$

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Covariance and Autocorrelation

Given a two random variables X and Y .

Covariance

$$\mu_{XY} = \mathbb{E} \{ [X - \bar{x}][Y - \bar{y}] \} = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \quad (19)$$

Correlation Coefficient

$$\rho_{XY} = \frac{\mu_{XY}}{\sigma_X \sigma_Y} \quad (20)$$

Autocorrelation

$$\Gamma_X(\tau) = \mathbb{E}[X(t)X(t + \tau)] \quad (21)$$

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2.2 Stochastic Processes

Terminology

See Figure 1

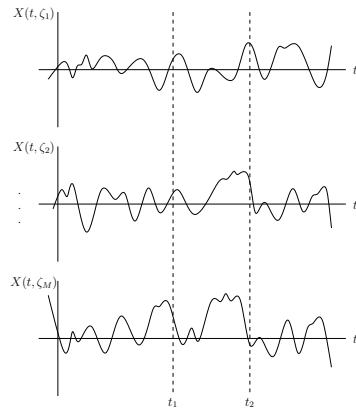


Figure 1: Sample functions of a random process

- $X(t, \zeta_i)$: sample function.
- The governing experiment: random or stochastic process.
- All sample functions: ensemble.
- $X(t_j, \zeta)$: random variable.

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Strict Sense Stationarity

If the joint pdfs depend only on the time difference regardless of the time origin, then the random process is known as *stationary*.

For stationary process means and variances are independent of time and the covariance depends only on the time difference.

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Wide Sense Stationarity

If the joint pdfs depends on the time difference but the mean and variances are time-independent, then the random process is known as *wide-sense-stationary*.

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Ergodicity

If the time statistics equals ensemble statistics, then the random process is known as *ergodic*.

Any statistic calculated by averaging of all members of an ergodic ensemble at a fixed time can also be calculated by using a single representative waveform and averaging over all time.

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2.3 Correlation and Power Spectral Density

Power Spectral Density

Given a sample function $X(t, \zeta_i)$ of a random process, we obtain the power spectral density by

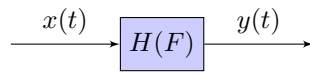
$$S(F) \xleftrightarrow{\mathcal{F}} \Gamma(\tau) \quad (22)$$

i.e., for a wide sense stationary signal, the power spectral density and autocorrelation are Fourier transform pairs.

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2.4 Input-Output Relationship of Linear Systems

Input-Output Relationship of Linear Systems



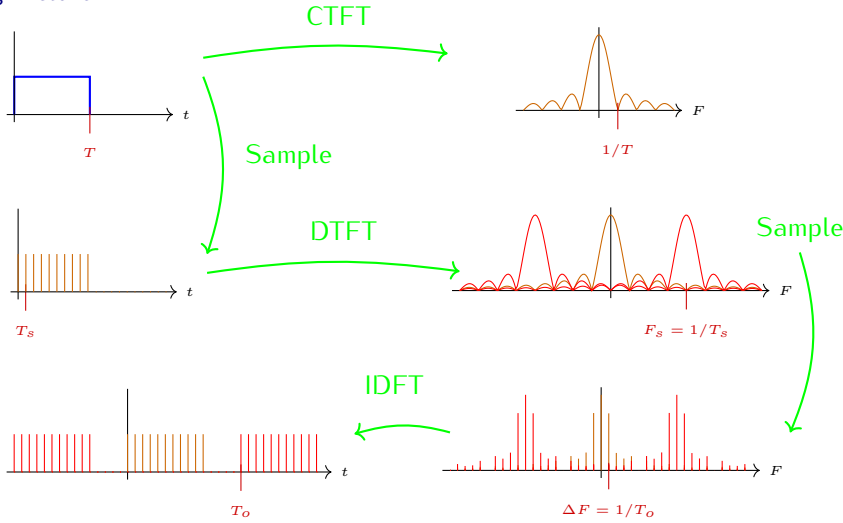
$$S_Y(F) = |H(F)|^2 S_X(F) \quad (23)$$

Noise Shaping

If $x(t)$ is white noise, we can design the filter $h(t)$ to “shape” the noise.

3 Discrete Signals and Systems

Big Picture



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Sampling Remarks

- Must sample more than twice bandwidth to avoid aliasing.
- FFT represents a periodic version of the time domain signal → could have time domain aliasing.
- Number of points in FFT is the same as number of points in time domain signal.

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4 Power Spectral Density

Obtaining PSD for Discrete Signals

What we want is

$$\Gamma_X(\tau) = \mathbb{E}[X(t)X(t + \tau)] \xrightarrow{\text{CTFT}} S_X(F)$$

For infinitely long signals.

What we can compute is

$$\gamma_X(m) = \mathbb{E}[X(n)X(n + m)] \xrightarrow{\text{DFT}} P_X(f)$$

For finite length signals.

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What do we need in an estimate

As $N \rightarrow \infty$ and in the mean squared sense

Unbiased

Asymptotically the mean of the estimate approaches the true power.

Variance

Variance of the estimate approaches zero.

Resulting in a consistent estimate of the power spectrum.

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Possible PSD Options

Periodogram

computed using $1/N$ times the magnitude squared of the FFT

$$\lim_{N \rightarrow \infty} \mathbb{E}[P_X(f)] = S_X(f)$$

$$\lim_{N \rightarrow \infty} \text{var}[P_X(f)] = S_X^2(f)$$

Welch Method

computed by segmenting the data (allowing overlaps), windowing the data in each segment then computing the average of the resultant periodogram

$$\mathbb{E}[P_X(f)] = \frac{1}{2\pi MU} S_X(f) \otimes W(f)$$

$$\text{var}[P_X(f)] \approx \frac{9}{8L} S_X^2(f)$$

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Welch Method

Assuming data length N , segment length M , Bartlett window, and 50% overlap

- FFT length = $M = 1.28/\Delta f = 1.28F_s/\Delta F$
- Resulting number of segments = $L = \frac{2N}{M}$
- Length of data collected in sec. = $\frac{1.28L}{2\Delta F}$

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pwelch Function

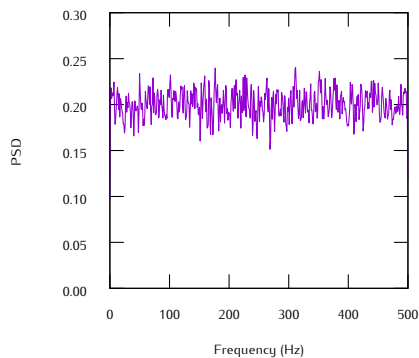
```
[Pxx,f] = pwelch(x,window,noverlap,...  
               nfft,fs,'range')
```

You can use `[]` in fields that you want the default to be used.

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pwelch Function - WGN signal

```
Fs = 1000;  
x = sqrt(0.1*Fs)*randn(1,100000);  
[Pxx,f] = pwelch(x,1024,[],[],Fs,'onesided');
```

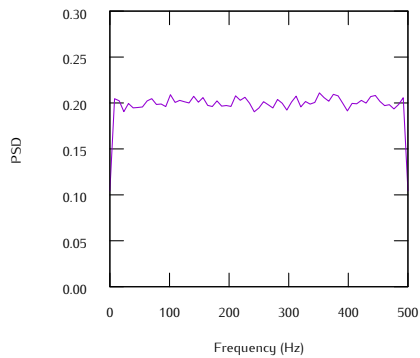


- Variance to high.

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pwelch Function - WGN signal

```
[Pxx,f] = pwelch(x,128,[],[],Fs,'onesided')
```

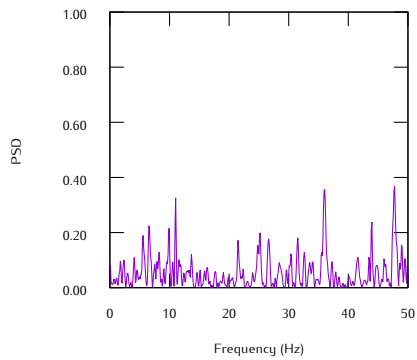


- Reduced window size.
- Variance is now smaller.

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pwelch Function - cos + WGN signal

```
Fs = 100; t = 0:1/Fs:5;  
x = cos(2*pi*10*t)+cos(2*pi*11*t)+...  
    sqrt(0.1*Fs)*randn(1,length(t));  
[Pxx,f] = pwelch(x,1024,[],[],Fs,'onesided');
```

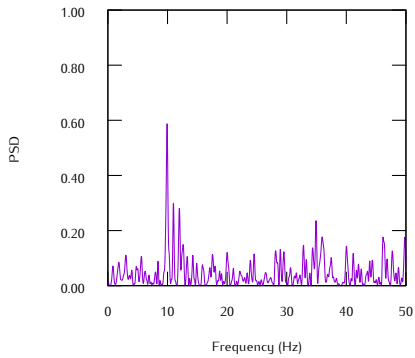


- Window larger than length of data.
- Frequency components can't be resolved.
- Variance high.

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pwelch Function - cos + WGN signal

```
Fs = 100; t = 0:1/Fs:5;  
x = cos(2*pi*10*t)+cos(2*pi*11*t)+...  
    sqrt(0.1*Fs)*randn(1,length(t));  
[Pxx,f] = pwelch(x,1024,[],4096,Fs,'onesided');
```



- As expected increasing `nFFT` does not help.

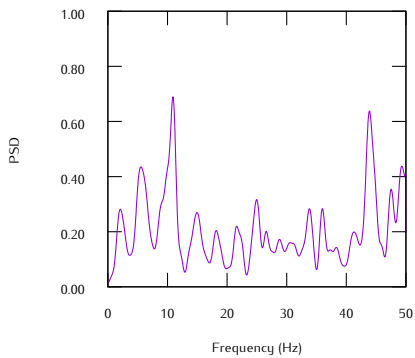
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pwelch Function - cos + WGN signal

```

Fs = 100;    t = 0:1/Fs:5;
x = cos(2*pi*10*t)+cos(2*pi*11*t)+...
    sqrt(0.1*Fs)*randn(1,length(t));
[Pxx,f] = pwelch(x,128,[],4096,Fs,'onesided');

```



- Decreasing the window size decreases the variance.
- Still can't resolve the two frequencies.

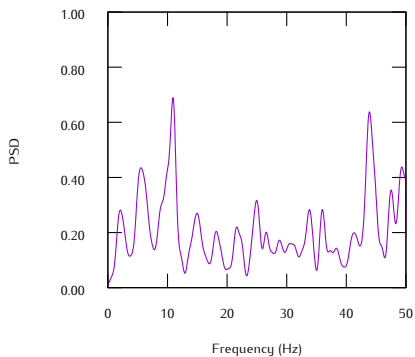
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pwelch Function - cos + WGN signal

```

Fs = 100;    t = 0:1/Fs:50;
x = cos(2*pi*10*t)+cos(2*pi*11*t)+...
    sqrt(0.1*Fs)*randn(1,length(t));
[Pxx,f] = pwelch(x,128,[],4096,Fs,'onesided');

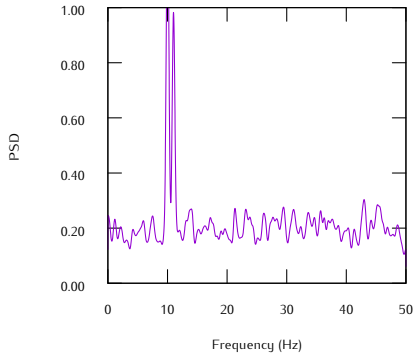
```



- Length of data sequence must be increased.
- Still can't resolve the two frequencies as the window size is too small.

pwelch Function - cos + WGN signal

```
Fs = 100;    t = 0:1/Fs:50;  
x = cos(2*pi*10*t)+cos(2*pi*11*t)+...  
    sqrt(0.1*Fs)*randn(1,length(t));  
[Pxx,f] = pwelch(x,256,[],4096,Fs,'onesided');
```



- Now we can resolve the two frequencies.

Spectral Estimation - Remarks

- The length of the data sequence determines the maximum resolution that can be observed.
- Increasing the window length of each segment in the data increases the resolution.
- Decreasing the window length of each segment in the data decreases the variance of the estimate.
- `nFFT` only affects the amount of details shown and not the resolution.