EE 565: Position, Navigation and Timing Error Mechanization (Tangential)

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April 9, 2023

Tangential Mechanization — Continuous Case



In continuous time notation

• Attitude:
$$C_b^t = C_b^t \Omega_{tb}^b$$

• Velocity:
$$\dot{\vec{v}}_{tb}^t = C_b^t \vec{f}_{ib}^b + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^t$$

- Position: $\dot{\vec{r}}_{ib}^t = \vec{v}_{tb}^t$
- In State-space notation

$$\Omega^b_{tb} = \Omega^b_{ib} - \Omega^b_{ie}$$

$$\vec{\omega}_{ib}^b = \vec{\omega}_{ie}^b + \vec{\omega}_{tb}^b$$

$$\begin{bmatrix} \vec{r}_{tb}^t \\ \vec{v}_{tb}^t \\ \dot{C}_b^t \end{bmatrix} = \begin{bmatrix} \vec{v}_{tb}^t \\ C_b^t \vec{f}_{ib}^b + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^t \\ C_b^t \Omega_{tb}^b \end{bmatrix}$$

(1)

Attitude Velocity Gravity Position Summary

Tangential Mechanization — Continuous Case



In continuous time notation

• Attitude:
$$\dot{C}_b^t = C_b^t \Omega_{\underline{t}b}^b$$

• Velocity:
$$\dot{\vec{v}}_{tb}^t = C_b^t \vec{f}_{ib}^b + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^t$$

• Position:
$$\dot{\vec{r}}_{ib}^t = \vec{v}_{tb}^t$$

• In State-space notation

$$\Omega^b_{tb} = \Omega^b_{ib} - \Omega^b_{ie}$$

$$\vec{\omega}_{ib}^b = \vec{\omega}_{ie}^b + \vec{\omega}_{tb}^b$$

$$\begin{bmatrix} \dot{\vec{r}}_{tb}^t \\ \dot{\vec{v}}_{tb}^t \\ \dot{C}_b^t \end{bmatrix} = \begin{bmatrix} \vec{v}_{tb}^t \\ C_b^t \vec{f}_{ib}^b + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^t \\ C_b^t \Omega_{tb}^b \end{bmatrix}$$
(1)

Question

What is the effect of sensor noise in the measurements $\vec{\omega}_{ib}^b$ and \vec{f}_{ib}^b on the navigation solution position, velocity and attitude?

Attitude

Velocity

Gravity

Positio

Summaru



$$\dot{C}_b^t = C_b^t \Omega_{tb}^b = C_b^t (\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt} \left[(\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \right] = 0$$



$$\dot{C}_b^t = C_b^t \Omega_{tb}^b = C_b^t (\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt} \left[(\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \right] =$$

$$(\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \Omega_{tb}^b = [\delta \dot{\vec{\psi}}_{tb}^t \times] \hat{C}_b^t + (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \dot{\hat{C}}_b^t =$$



$$\dot{C}^t_b = C^t_b \Omega^b_{tb} = C^t_b (\Omega^b_{ib} - \Omega^b_{ie}) = \frac{d}{dt} \left[(\mathcal{I} + [\delta \vec{\psi}^t_{tb} \times]) \hat{C}^t_b \right] =$$

$$(\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \Omega_{tb}^b = [\delta \vec{\psi}_{tb}^t \times] \hat{C}_b^t + (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t = (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t (\hat{\Omega}_{tb}^b + \delta \Omega_{ib}^b - \delta \Omega_{ie}^b) =$$



$$\dot{C}^t_b = C^t_b \Omega^b_{tb} = C^t_b (\Omega^b_{ib} - \Omega^b_{ie}) = \frac{d}{dt} \left[(\mathcal{I} + [\delta \vec{\psi}^t_{tb} \times]) \hat{C}^t_b \right] =$$

$$(\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \Omega_{tb}^b = [\delta \dot{\psi}_{tb}^t \times] \hat{C}_b^t + (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \dot{\hat{C}}_b^t =$$

$$(\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t (\hat{\Omega}_{tb}^b + \delta \Omega_{ib}^b - \delta \Omega_{ie}^b) =$$

$$(\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \hat{\Omega}_{tb}^b + \hat{C}_b^t (\delta \Omega_{ib}^b - \delta \Omega_{ie}^b) =$$

$$(\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \hat{\Omega}_{tb}^b + \hat{C}_b^t (\delta \Omega_{ib}^b - \delta \Omega_{ie}^b) =$$

$$(\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \hat{\Omega}_{tb}^b \times \hat{C}_b^t (\delta \Omega_{ib}^b - \delta \Omega_{ie}^b) =$$

$$(\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \hat{\Omega}_{tb}^b \times \hat{C}_b^t (\delta \Omega_{ib}^b - \delta \Omega_{ie}^b) =$$

$$(\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \hat{\Omega}_{tb}^b \times \hat{C}_b^t (\delta \Omega_{ib}^b - \delta \Omega_{ie}^b) =$$

Attitude Velocity Gravity Position Summary

O O O O O O O



$$\dot{C}_{b}^{t} = C_{b}^{t}\Omega_{tb}^{b} = C_{b}^{t}(\Omega_{ib}^{b} - \Omega_{ie}^{b}) = \frac{d}{dt} \left[(\mathcal{I} + [\delta\vec{\psi}_{tb}^{t} \times])\hat{C}_{b}^{t} \right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{tb}^{t} \times])\hat{C}_{b}^{t}\Omega_{tb}^{b} = [\delta\dot{\psi}_{tb}^{t} \times]\hat{C}_{b}^{t} + (\mathcal{I} + [\delta\vec{\psi}_{tb}^{t} \times])\hat{C}_{b}^{t}) =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{tb}^{t} \times])\hat{C}_{b}^{t}(\hat{\Omega}_{tb}^{b} + \delta\Omega_{ib}^{b} - \delta\Omega_{ie}^{b}) =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{tb}^{t} \times])\hat{C}_{b}^{t}(\hat{\Omega}_{tb}^{b} + \hat{C}_{b}^{t}(\delta\Omega_{ib}^{b} - \delta\Omega_{ie}^{b}) =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{tb}^{t} \times])\hat{C}_{b}^{t}\hat{\Omega}_{tb}^{b} + \hat{C}_{b}^{t}(\delta\Omega_{ib}^{b} - \delta\Omega_{ie}^{b}) =$$



$$\dot{C}^t_b = C^t_b \Omega^b_{tb} = C^t_b (\Omega^b_{ib} - \Omega^b_{ie}) = rac{d}{dt} \left[(\mathcal{I} + [\delta \vec{\psi}^t_{tb} imes]) \hat{C}^t_b
ight] =$$

$$(\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \Omega_{tb}^b = [\delta \dot{\vec{\psi}}_{tb}^t \times] \hat{C}_b^t + (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t = (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t (\hat{\Omega}_{tb}^b + \delta \Omega_{ib}^b - \delta \Omega_{ie}^b) = (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \hat{\Omega}_{tb}^b + \hat{C}_b^t (\delta \Omega_{ib}^b - \delta \Omega_{ie}^b) =$$

$$[\delta \vec{\psi}_{tb}^{t} \times] = \hat{C}_{b}^{t} (\delta \Omega_{ib}^{b} - \delta \Omega_{ie}^{b}) \hat{C}_{t}^{b} = [\hat{C}_{b}^{t} (\delta \vec{\omega}_{ib}^{b} - \delta \vec{\omega}_{ie}^{b}) \times]$$
(2)

Attitude Velocity Gravity Position Summary



$$\dot{C}^t_b = C^t_b \Omega^b_{tb} = C^t_b (\Omega^b_{ib} - \Omega^b_{ie}) = rac{d}{dt} \left[(\mathcal{I} + [\delta \vec{\psi}^t_{tb} imes]) \hat{C}^t_b
ight] =$$

$$\begin{aligned} (\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\hat{C}_b^t\Omega_{tb}^b &= [\delta\dot{\psi}_{tb}^t \times]\hat{C}_b^t + (\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\dot{\hat{C}}_b^t = \\ (\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\hat{C}_b^t(\hat{\Omega}_{tb}^b + \delta\Omega_{ib}^b - \delta\Omega_{ie}^b) &= \\ (\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\hat{C}_b^t\hat{\Omega}_{tb}^b + \hat{C}_b^t(\delta\Omega_{ib}^b - \delta\Omega_{ie}^b) &= \end{aligned}$$

$$[\delta \vec{\psi}_{tb}^t \times] = \hat{C}_b^t (\delta \Omega_{ib}^b - \delta \Omega_{ie}^b) \hat{C}_t^b = [\hat{C}_b^t (\delta \vec{\omega}_{ib}^b - \delta \vec{\omega}_{ie}^b) \times]$$
(2)

$$\delta \vec{\psi}_{tb}^t = \hat{C}_b^t (\delta \vec{\omega}_{ib}^b - \delta \vec{\omega}_{ie}^b) \tag{3}$$

Attitude

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Tangential Attitude Error (cont.)



$$\begin{split} \delta \dot{\vec{\psi}}_{tb}^t &= \hat{C}_b^t (\delta \vec{\omega}_{ib}^b - \delta \vec{\omega}_{ie}^b) \\ &= \hat{C}_b^t \delta \vec{\omega}_{ib}^b - \hat{C}_b^t (\vec{\omega}_{ie}^b - \hat{\vec{\omega}}_{ie}^b) \\ &= \hat{C}_b^t \delta \vec{\omega}_{ib}^b - (\hat{C}_b^t C_t^b - \mathcal{I}) \vec{\omega}_{ie}^t \\ &= \hat{C}_b^t \delta \vec{\omega}_{ib}^b + \delta \vec{\psi}_{tb}^t \times \vec{\omega}_{ie}^t \end{split}$$

Tangential Attitude Error (cont.)



$$\begin{split} \delta \dot{\vec{\psi}}_{tb}^t &= \hat{C}_b^t (\delta \vec{\omega}_{ib}^b - \delta \vec{\omega}_{ie}^b) \\ &= \hat{C}_b^t \delta \vec{\omega}_{ib}^b - \hat{C}_b^t (\vec{\omega}_{ie}^b - \hat{\vec{\omega}}_{ie}^b) \\ &= \hat{C}_b^t \delta \vec{\omega}_{ib}^b - (\hat{C}_b^t C_t^b - \mathcal{I}) \vec{\omega}_{ie}^t \\ &= \hat{C}_b^t \delta \vec{\omega}_{ib}^b + \delta \vec{\psi}_{tb}^t \times \vec{\omega}_{ie}^t \end{split}$$

$$\delta \dot{\vec{\psi}}_{tb}^t = \hat{C}_b^t \delta \vec{\omega}_{ib}^b - \vec{\Omega}_{ie}^t \delta \vec{\psi}_{tb}^t$$

Attitude



$$\dot{\vec{v}}_{tb}^{t} = C_{b}^{t} \vec{f}_{ib}^{b} + \vec{g}_{b}^{t} - 2\Omega_{ie}^{t} \vec{v}_{tb}^{t}$$
(5)

$$\dot{\vec{v}}_{tb}^{t} = \hat{C}_{b}^{t} \hat{\vec{f}}_{ib}^{b} + \hat{\vec{g}}_{b}^{t} - 2\Omega_{ie}^{t} \hat{\vec{v}}_{tb}^{t}
= (\mathcal{I} - [\delta \vec{\psi}_{tb}^{t} \times]) C_{b}^{t} (\vec{f}_{ib}^{b} - \delta \vec{f}_{ib}^{b}) + \hat{\vec{g}}_{b}^{t} - 2\Omega_{ie}^{t} \hat{\vec{v}}_{tb}^{t}$$
(6)



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$$\dot{\vec{v}}_{tb}^{t} = C_{b}^{t} \vec{f}_{ib}^{b} + \vec{g}_{b}^{t} - 2\Omega_{ie}^{t} \vec{v}_{tb}^{t} \tag{5}$$

$$\dot{\vec{v}}_{tb}^{t} = \hat{C}_{b}^{t} \hat{\vec{f}}_{ib}^{b} + \hat{\vec{g}}_{b}^{t} - 2\Omega_{ie}^{t} \hat{\vec{v}}_{tb}^{t}
= (\mathcal{I} - [\delta \vec{\psi}_{tb}^{t} \times]) C_{b}^{t} (\vec{f}_{ib}^{b} - \delta \vec{f}_{ib}^{b}) + \hat{\vec{g}}_{b}^{t} - 2\Omega_{ie}^{t} \hat{\vec{v}}_{tb}^{t}$$
(6)

$$\begin{split} \delta \vec{v}_{tb}^t &= \dot{\vec{v}}_{tb}^t - \dot{\hat{\vec{v}}}_{tb}^t = [\delta \vec{\psi}_{tb}^t \times] C_b^t \vec{f}_{ib}^b + \hat{C}_b^t \delta \vec{f}_{ib}^b + \delta \vec{g}_b^t - 2\Omega_{ie}^t \delta \vec{v}_{tb}^t \\ &= [\delta \vec{\psi}_{tb}^t \times] \hat{C}_b^t \hat{\vec{f}}_{ib}^b + \hat{C}_b^t \delta \vec{f}_{ib}^b + \delta \vec{g}_b^t - 2\Omega_{ie}^t \delta \vec{v}_{tb}^t \end{split}$$



$$\dot{\vec{v}}_{tb}^{t} = C_b^t \vec{f}_{ib}^{b} + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^t$$
 (5)

$$\dot{\vec{v}}_{tb}^{t} = \hat{C}_{b}^{t} \vec{f}_{ib}^{b} + \hat{\vec{g}}_{b}^{t} - 2\Omega_{ie}^{t} \hat{\vec{v}}_{tb}^{t}
= (\mathcal{I} - [\delta \vec{\psi}_{tb}^{t} \times]) C_{b}^{t} (\vec{f}_{ib}^{b} - \delta \vec{f}_{ib}^{b}) + \hat{\vec{g}}_{b}^{t} - 2\Omega_{ie}^{t} \hat{\vec{v}}_{tb}^{t}$$
(6)

$$\begin{split} \delta \vec{\mathbf{v}}_{tb}^t &= \dot{\vec{\mathbf{v}}}_{tb}^t - \dot{\hat{\vec{\mathbf{v}}}}_{tb}^t = [\delta \vec{\psi}_{tb}^t \times] C_b^t \vec{f}_{ib}^b + \hat{C}_b^t \delta \vec{f}_{ib}^b + \delta \vec{g}_b^t - 2\Omega_{ie}^t \delta \vec{\mathbf{v}}_{tb}^t \\ &= [\delta \vec{\psi}_{tb}^t \times] \hat{C}_b^t \hat{f}_{ib}^b + \hat{C}_b^t \delta \vec{f}_{ib}^b + \delta \vec{g}_b^t - 2\Omega_{ie}^t \delta \vec{\mathbf{v}}_{tb}^t \end{split}$$

$$\delta \vec{\mathbf{v}}_{tb}^{t} = -[\hat{C}_{b}^{t}\hat{\vec{f}}_{ib}^{b} \times]\delta \vec{\psi}_{tb}^{t} + \hat{C}_{b}^{t}\delta \vec{f}_{ib}^{b} + \delta \vec{\mathbf{g}}_{b}^{t} - 2\Omega_{ie}^{t}\hat{\vec{\mathbf{v}}}_{tb}^{t}$$

$$(7)$$

Attitude Velocity Gravity Position Summary

$$\delta \vec{g}_b^t \approx \hat{C}_e^t \frac{2g_0(\hat{L}_b)}{r_{eS}^e(\hat{L}_b)} \frac{\hat{\vec{r}}_{eb}^e}{|\hat{\vec{r}}_{eb}^e|^2} (\hat{\vec{r}}_{eb}^e)^T \hat{C}_t^e \delta \vec{r}_{tb}^t$$
(8)



$$\dot{\vec{r}}_{tb}^{t} = \vec{v}_{tb}^{t} \tag{9}$$

$$\dot{\vec{r}}_{tb}^{t} = \vec{v}_{tb}^{t}$$

$$\delta \dot{\vec{r}}_{tb}^{t} = \delta \vec{v}_{tb}^{t} \tag{10}$$

(9)

April 9, 2023

Summary - in terms of $\delta ec{f}_{ib}^{\ b}, \delta ec{\omega}_{ib}^{\ b}$



$$\begin{pmatrix}
\delta \vec{\psi}_{tb}^{t} \\
\delta \vec{v}_{tb}^{t} \\
\delta \vec{r}_{tb}^{t}
\end{pmatrix} = \begin{bmatrix}
-\Omega_{ie}^{t} & 0_{3\times3} & 0_{3\times3} \\
-[\hat{C}_{b}^{t}\hat{\vec{f}}_{ib}^{b}\times] & -2\Omega_{ie}^{t} & \hat{C}_{e}^{t}\frac{2g_{0}(\hat{L}_{b})}{r_{es}^{e}(\hat{L}_{b})}\frac{\hat{r}_{eb}^{e}}{|\hat{r}_{eb}^{e}|^{2}}(\hat{r}_{eb}^{e})^{T}\hat{C}_{t}^{e} \\
0_{3\times3} & \mathcal{I}_{3\times3} & 0_{3\times3}
\end{bmatrix} \begin{pmatrix}
\delta \vec{\psi}_{tb}^{t} \\
\delta \vec{r}_{tb}^{t}
\end{pmatrix} + \begin{pmatrix}
0 & \hat{C}_{b}^{t} \\
\hat{C}_{b}^{t} & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
\delta \vec{f}_{ib}^{b} \\
\delta \vec{\omega}_{ib}^{b}
\end{pmatrix} \tag{11}$$

Attitude Velocity Gravity Position **Summary**oo o o •



Truth value

$$\vec{x}$$

Measured value

Estimated or computed value

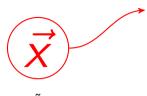
$$\hat{\vec{x}}$$

Error

$$\delta \vec{x} = \vec{x} - \hat{\vec{x}}$$



- Truth value
- Measured value
- Estimated or computed value
- Error





 $\hat{\vec{x}}$

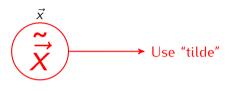
$$\delta \vec{x} = \vec{x} - \hat{\vec{x}}$$



- Truth value
- Measured value

• Estimated or computed value

Error



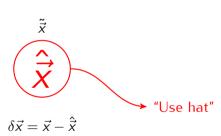


$$\delta \vec{x} = \vec{x} - \hat{\vec{x}}$$



- Truth value
- Measured value
- Estimated or computed value

Error



 \vec{x}



Truth value



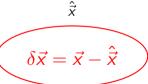
Measured value



Estimated or computed value



Error





Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$



Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$

Let's assume we have an estimate of \vec{x} , i.e., $\hat{\vec{x}}$ such that $\vec{x} = \hat{\vec{x}} + \delta \vec{x}$

$$\dot{ec{x}}=\dot{ec{x}}+\delta\dot{ec{x}}=f(\hat{ec{x}}+\deltaec{x},t)$$



Given a non-linear system $\vec{x} = f(\vec{x}, t)$

Let's assume we have an estimate of \vec{x} , i.e., $\hat{\vec{x}}$ such that $\vec{x} = \hat{\vec{x}} + \delta \vec{x}$

$$\dot{\vec{x}} = \dot{\vec{x}} + \delta \dot{\vec{x}} = f(\hat{\vec{x}} + \delta \vec{x}, t)$$
 (12)

Using Taylor series expansion

$$f(\hat{\vec{x}} + \delta \vec{x}, t) = \dot{\hat{\vec{x}}} + \delta \dot{\vec{x}} = f(\hat{\vec{x}}, t) + \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \Big|_{\vec{x} = \hat{\vec{x}}} \delta \vec{x} + H.O.T$$

$$\approx \dot{\hat{\vec{x}}} + \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \Big|_{\vec{x} = \hat{\vec{x}}} \delta \vec{x}$$



Given a non-linear system $\vec{x} = f(\vec{x}, t)$

Let's assume we have an estimate of \vec{x} , i.e., $\hat{\vec{x}}$ such that $\vec{x} = \hat{\vec{x}} + \delta \vec{x}$

$$\dot{\vec{x}} = \dot{\vec{x}} + \delta \dot{\vec{x}} = f(\hat{\vec{x}} + \delta \vec{x}, t)$$
 (12)

Using Taylor series expansion

$$f(\hat{\vec{x}} + \delta \vec{x}, t) = \dot{\hat{\vec{x}}} + \delta \dot{\vec{x}} = \underbrace{f(\hat{\vec{x}}, t)}_{\vec{x}} + \underbrace{\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}}_{\vec{x} = \hat{\vec{x}}} \delta \vec{x} + H.O.T$$

$$\approx \dot{\hat{\vec{x}}} + \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \Big|_{\vec{x} = \hat{\vec{x}}} \delta \vec{x}$$



Given a non-linear system $\vec{x} = f(\vec{x}, t)$

Let's assume we have an estimate of \vec{x} , i.e., $\hat{\vec{x}}$ such that $\vec{x} = \hat{\vec{x}} + \delta \vec{x}$

$$\dot{\vec{x}} = \dot{\vec{x}} + \delta \dot{\vec{x}} = f(\hat{\vec{x}} + \delta \vec{x}, t)$$
 (12)

Using Taylor series expansion

$$f(\hat{\vec{x}} + \delta \vec{x}, t) = \dot{\hat{\vec{x}}} + \delta \dot{\vec{x}} = f(\hat{\vec{x}}, t) + \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \Big|_{\vec{x} = \hat{\vec{x}}} \delta \vec{x} + H.O.T$$

$$\approx \dot{\hat{\vec{x}}} + \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \Big|_{\vec{x} = \hat{\vec{x}}} \delta \vec{x}$$

$$\Rightarrow \delta \dot{\vec{x}} \approx \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x} = \hat{\vec{x}}} \delta \vec{x} \tag{13}$$

Actual Measurements



Initially the accelerometer and gyroscope measurements, $\tilde{\vec{f}}_{ib}^{b}$ and $\tilde{\vec{\omega}}_{ib}^{b}$, respectively, will be modeled as

$$\tilde{\vec{f}}_{ib}^b = \vec{f}_{ib}^b + \Delta \vec{f}_{ib}^b \tag{14}$$

$$\tilde{\vec{\omega}}_{ib}^{b} = \vec{\omega}_{ib}^{b} + \Delta \vec{\omega}_{ib}^{b} \tag{15}$$

where $\vec{f}_{ib}^{\ b}$ and $\vec{\omega}_{ib}^{\ b}$ are the specific force and angular rates, respectively; and $\Delta \vec{f}_{ib}^{\ b}$ and $\Delta \vec{\omega}_{ib}^{\ b}$ represents the errors. In later lectures we will discuss more detailed description of these errors.

Actual Measurements



Initially the accelerometer and gyroscope measurements, \tilde{f}^b_{ib} and $\tilde{\omega}^b_{ib}$, respectively, will be modeled as

$$\tilde{\vec{f}}_{ib}^{b} = \vec{f}_{ib}^{b} + \Delta \vec{f}_{ib}^{b}$$
these terms may
$$\tag{14}$$

$$\tilde{\vec{\omega}}_{ib}^{b} = \vec{\omega}_{ib}^{b} + \left(\Delta \vec{\omega}_{ib}^{b}\right)^{\text{be expanded further}} \tag{15}$$

where $\vec{f}_{ib}^{\ b}$ and $\vec{\omega}_{ib}^{\ b}$ are the specific force and angular rates, respectively; and $\Delta \vec{f}_{ib}^{\ b}$ and $\Delta \vec{\omega}_{ib}^{\ b}$ represents the errors. In later lectures we will discuss more detailed description of these errors.



Accelerometers

$$\tilde{\vec{f}}_{ib}^{b} = \vec{b}_{a} + (\mathcal{I} + M_{a})\vec{f}_{ib}^{b} + \vec{nl}_{a} + \vec{w}_{a}$$
 (16)

Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^{b} = \vec{b}_{g} + (\mathcal{I} + M_{g})\vec{\omega}_{ib}^{b} + G_{g}\vec{f}_{ib}^{b} + \vec{w}_{g}$$
(17)

Basic Definitions

earization

Inertial Measurements

Attitude Error

Estimates of Sensor Measurements

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Accelerometers

$$\tilde{\vec{f}}_{ib}^{b} = \vec{\vec{b}}_{a} + (\mathcal{I} + M_{a})\vec{f}_{ib}^{b} + n\vec{I}_{a} + \vec{w}_{a}$$
Biases
$$\tilde{\vec{\omega}}_{ib}^{b} = \vec{\vec{b}}_{g} + (\mathcal{I} + M_{g})\vec{\omega}_{ib}^{b} + G_{g}\vec{f}_{ib}^{b} + \vec{w}_{g}$$

Gyroscopes

 Inertial Measurements
 Attitude Error
 Estimates of Sensor Measurements

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Accelerometers

$$\tilde{\vec{f}}_{ib}^{b} = \vec{b}_{a} + (\mathcal{I} + M_{a})\vec{f}_{ib}^{b} + \vec{nl}_{a} + \vec{w}_{a}$$
Misalignment and SF Errors
$$\tilde{\vec{\omega}}_{ib}^{b} = \vec{b}_{g} + (\mathcal{I} + M_{g})\vec{\omega}_{ib}^{b} + G_{g}\vec{f}_{ib}^{b} + \vec{w}_{g}$$

(16)

(17)

Gyroscopes



Accelerometers

$$\vec{f}_{ib}^{\,b} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^{\,b} + \vec{nl}_a + \vec{w}_a$$

Non-linearity

(16)

(17)

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Gyroscopes

$$ilde{ec{\omega}}_{ib}^{\,b} = ec{b}_{\mathcal{g}} + (\mathcal{I} + M_{\mathcal{g}}) ec{\omega}_{ib}^{\,b} + G_{\mathcal{g}} ec{f}_{ib}^{\,b} + ec{w}_{\mathcal{g}}$$

EE 565: Position, Navigation and Timing



Accelerometers

$$\tilde{\vec{f}}_{ib}^{b} = \vec{b}_{a} + (\mathcal{I} + M_{a})\vec{f}_{ib}^{b} + \vec{nl}_{a} + \vec{w}_{a}$$
 (16)

 $ilde{ec{\omega}}_{ib}^{\,b} = ec{b}_{oldsymbol{g}} + (\mathcal{I} + M_{oldsymbol{g}})ec{\omega}_{\,ib}^{\,b} + oldsymbol{G}_{oldsymbol{g}}ec{f}_{\,ib}^{\,b} + ec{w}_{oldsymbol{g}}$

G-Sensitivity

Gyroscopes

(17)



(16)

Accelerometers

$$\tilde{\vec{f}}_{ib}^{b} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^{b} + \vec{nl}_a + \vec{w}_a$$
Noise

Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^{b} = \vec{b}_{g} + (\mathcal{I} + M_{g})\vec{\omega}_{ib}^{b} + G_{g}\vec{f}_{ib}^{b} + \vec{w}_{g}$$

(17)

Pos, Vel, Force and Angular Rate Errors



Position error

$$\delta \vec{r}_{\beta b}^{\gamma} = \vec{r}_{\beta b}^{\gamma} - \hat{\vec{r}}_{\beta b}^{\gamma} \tag{18}$$

Velocity error

$$\delta \vec{v}_{\beta b}^{\gamma} = \vec{v}_{\beta b}^{\gamma} - \hat{\vec{v}}_{\beta b}^{\gamma} \tag{19}$$

Specific force errors

$$\delta \vec{f}_{ib}^{\ b} = \vec{f}_{ib}^{\ b} - \hat{\vec{f}}_{ib}^{\ b} \tag{20}$$

$$\Delta_e \vec{f}_{ib}^{\ b} = \Delta \vec{f}_{ib}^{\ b} - \Delta \hat{\vec{f}}_{ib}^{\ b} = -\delta \vec{f}_{ib}^{\ b}$$

Angular rate errors

$$\delta \vec{\omega}_{ib}^{\ b} = \vec{\omega}_{ib}^{\ b} - \hat{\vec{\omega}}_{ib}^{\ b} \tag{22}$$

$$\Delta_e \vec{\omega}_{ib}^{\ b} = \Delta \vec{\omega}_{ib}^{\ b} - \Delta \hat{\vec{\omega}}_{ib}^{\ b} = -\delta \vec{\omega}_{ib}^{\ b} \tag{23}$$

(21)

Attitude Error Definition



Define

$$\delta C_b^{\gamma} = C_b^{\gamma} \hat{C}_{\gamma}^b = e^{[\delta \vec{\psi}_{\gamma b}^{\gamma} \times]} \approx \mathcal{I} + [\delta \vec{\psi}_{\gamma b}^{\gamma} \times]$$
 (24)

This is the error in attitude resulting from errors in estimating the angular rates.

Attitude Error Properties



The attitude error is a multiplicative small angle transformation from the actual frame to the computed frame

$$\hat{C}_{b}^{\gamma} = (\mathcal{I} - [\delta \vec{\psi}_{\gamma b}^{\gamma} \times]) C_{b}^{\gamma}$$
(25)

Similarly,

$$C_b^{\gamma} = (\mathcal{I} + [\delta \vec{\psi}_{\gamma b}^{\gamma} \times]) \hat{C}_b^{\gamma}$$
 (26)

Specific Force and Agnular Rates



Similarly we can attempt to estimate the specific force and angular rate by applying correction based on our estimate of the error.

$$\hat{\vec{f}}_{ib}^{\ b} = \tilde{\vec{f}}_{ib}^{\ b} - \Delta \hat{\vec{f}}_{ib}^{\ b} \tag{27}$$

$$\hat{\vec{\omega}}_{ib}^{b} = \tilde{\vec{\omega}}_{ib}^{b} - \Delta \hat{\vec{\omega}}_{ib}^{b} \tag{28}$$

where $\hat{\vec{f}}_{ib}^{\ b}$ and $\hat{\vec{\omega}}_{ib}^{\ b}$ are the accelerometer and gyroscope estimated calibration values, respectively.