

EE 565: Position, Navigation and Timing

Error Mechanization (Tangential)

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- In continuous time notation
 - Attitude: $\dot{C}_b^t = C_b^t \Omega_{tb}^b$
 - Velocity: $\dot{\vec{v}}_{tb}^t = C_b^t \vec{f}_{ib}^b + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^t$
 - Position: $\dot{\vec{r}}_{ib}^t = \vec{v}_{tb}^t$
- In State-space notation

$$\Omega_{tb}^b = \Omega_{ib}^b - \Omega_{ie}^b$$

$$\vec{\omega}_{ib}^b = \vec{\omega}_{ie}^b + \vec{\omega}_{tb}^b$$

$$\begin{bmatrix} \dot{\vec{r}}_{tb}^t \\ \dot{\vec{v}}_{tb}^t \\ \dot{C}_b^t \end{bmatrix} = \begin{bmatrix} \vec{v}_{tb}^t \\ C_b^t \vec{f}_{ib}^b + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^t \\ C_b^t \Omega_{tb}^b \end{bmatrix} \quad (1)$$

- In continuous time notation
 - Attitude: $\dot{C}_b^t = C_b^t \Omega_{tb}^b$
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 - Position: $\dot{\vec{r}}_{ib}^t = \vec{v}_{tb}^t$
- In State-space notation

$$\Omega_{tb}^b = \Omega_{ib}^b - \Omega_{ie}^b$$

$$\vec{\omega}_{ib}^b = \vec{\omega}_{ie}^b + \vec{\omega}_{tb}^b$$

$$\begin{bmatrix} \dot{\vec{r}}_{tb}^t \\ \dot{\vec{v}}_{tb}^t \\ \dot{C}_b^t \end{bmatrix} = \begin{bmatrix} \vec{v}_{tb}^t \\ C_b^t \vec{f}_{ib}^b + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^t \\ C_b^t \Omega_{tb}^b \end{bmatrix} \quad (1)$$

Question

What is the effect of sensor noise in the measurements $\vec{\omega}_{ib}^b$ and \vec{f}_{ib}^b on the navigation solution position, velocity and attitude?

$$\dot{\hat{C}}_b^t = C_b^t \Omega_{tb}^b = C_b^t (\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt} \left[(\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \right] =$$

$$\dot{\hat{C}}_b^t = C_b^t \Omega_{tb}^b = C_b^t (\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt} \left[(\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \right] =$$

$$(\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \Omega_{tb}^b = [\delta \dot{\vec{\psi}}_{tb}^t \times] \hat{C}_b^t + (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \dot{\hat{C}}_b^t =$$

$$\dot{\hat{C}}_b^t = C_b^t \Omega_{tb}^b = C_b^t (\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt} \left[(\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \right] =$$

$$\begin{aligned} (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \Omega_{tb}^b &= [\delta \dot{\vec{\psi}}_{tb}^t \times] \hat{C}_b^t + (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \dot{\hat{C}}_b^t = \\ (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t (\hat{\Omega}_{tb}^b + \delta \Omega_{ib}^b - \delta \Omega_{ie}^b) &= \end{aligned}$$

$$\dot{\hat{C}}_b^t = C_b^t \Omega_{tb}^b = C_b^t (\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt} \left[(\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times]) \hat{C}_b^t \right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times]) \hat{C}_b^t \Omega_{tb}^b = [\delta\dot{\vec{\psi}}_{tb}^t \times] \hat{C}_b^t + (\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times]) \dot{\hat{C}}_b^t =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times]) \hat{C}_b^t (\hat{\Omega}_{tb}^b + \delta\Omega_{ib}^b - \delta\Omega_{ie}^b) =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times]) \hat{C}_b^t \hat{\Omega}_{tb}^b + \hat{C}_b^t (\delta\Omega_{ib}^b - \delta\Omega_{ie}^b) =$$

$$\because [\delta\vec{\psi}_{tb}^t \times] \delta\Omega_{tb}^b \approx 0$$

$$\dot{\hat{C}}_b^t = C_b^t \Omega_{tb}^b = C_b^t (\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt} \left[(\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \right] =$$

$$\begin{aligned} (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \Omega_{tb}^b &= [\delta \dot{\vec{\psi}}_{tb}^t \times] \hat{C}_b^t + (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \dot{\hat{C}}_b^t = \\ (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t (\hat{\Omega}_{tb}^b + \delta \Omega_{ib}^b - \delta \Omega_{ie}^b) &= \\ (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \hat{\Omega}_{tb}^b + \hat{C}_b^t (\delta \Omega_{ib}^b - \delta \Omega_{ie}^b) &= \end{aligned}$$

$$\dot{\hat{C}}_b^t = C_{b_{tb}}^t \Omega_{tb}^b = C_b^t (\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt} \left[(\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \right] =$$

$$\begin{aligned} (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \Omega_{tb}^b &= [\delta \dot{\vec{\psi}}_{tb}^t \times] \hat{C}_b^t + (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \dot{\hat{C}}_b^t = \\ (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t (\hat{\Omega}_{tb}^b + \delta \Omega_{ib}^b - \delta \Omega_{ie}^b) &= \\ (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \hat{\Omega}_{tb}^b + \hat{C}_b^t (\delta \Omega_{ib}^b - \delta \Omega_{ie}^b) &= \end{aligned}$$

$$[\delta \dot{\vec{\psi}}_{tb}^t \times] = \hat{C}_b^t (\delta \Omega_{ib}^b - \delta \Omega_{ie}^b) \hat{C}_b^b = [\hat{C}_b^t (\delta \vec{\omega}_{ib}^b - \delta \vec{\omega}_{ie}^b) \times] \quad (2)$$

$$\dot{\hat{C}}_b^t = C_{b_{tb}}^t \Omega_{tb}^b = C_b^t (\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt} \left[(\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \right] =$$

$$\begin{aligned} (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \Omega_{tb}^b &= [\delta \dot{\vec{\psi}}_{tb}^t \times] \hat{C}_b^t + (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \dot{\hat{C}}_b^t = \\ (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t (\hat{\Omega}_{tb}^b + \delta \Omega_{ib}^b - \delta \Omega_{ie}^b) &= \\ (\mathcal{I} + [\delta \vec{\psi}_{tb}^t \times]) \hat{C}_b^t \hat{\Omega}_{tb}^b + \hat{C}_b^t (\delta \Omega_{ib}^b - \delta \Omega_{ie}^b) &= \end{aligned}$$

$$[\delta \dot{\vec{\psi}}_{tb}^t \times] = \hat{C}_b^t (\delta \Omega_{ib}^b - \delta \Omega_{ie}^b) \hat{C}_b^b = [\hat{C}_b^t (\delta \vec{\omega}_{ib}^b - \delta \vec{\omega}_{ie}^b) \times] \quad (2)$$

$$\delta \dot{\vec{\psi}}_{tb}^t = \hat{C}_b^t (\delta \vec{\omega}_{ib}^b - \delta \vec{\omega}_{ie}^b) \quad (3)$$

$$\begin{aligned}
 \dot{\delta\psi}_{tb}^t &= \hat{C}_b^t(\delta\vec{\omega}_{ib}^b - \delta\vec{\omega}_{ie}^b) \\
 &= \hat{C}_b^t\delta\vec{\omega}_{ib}^b - \hat{C}_b^t(\vec{\omega}_{ie}^b - \hat{\vec{\omega}}_{ie}^b) \\
 &= \hat{C}_b^t\delta\vec{\omega}_{ib}^b - (\hat{C}_b^t C_t^b - \mathcal{I})\vec{\omega}_{ie}^t \\
 &= \hat{C}_b^t\delta\vec{\omega}_{ib}^b + \delta\psi_{tb}^t \times \vec{\omega}_{ie}^t
 \end{aligned}$$

$$\begin{aligned}
 \delta \dot{\vec{\psi}}_{tb}^t &= \hat{C}_b^t (\delta \vec{\omega}_{ib}^b - \delta \vec{\omega}_{ie}^b) \\
 &= \hat{C}_b^t \delta \vec{\omega}_{ib}^b - \hat{C}_b^t (\vec{\omega}_{ie}^b - \hat{\vec{\omega}}_{ie}^b) \\
 &= \hat{C}_b^t \delta \vec{\omega}_{ib}^b - (\hat{C}_b^t C_t^b - \mathcal{I}) \vec{\omega}_{ie}^t \\
 &= \hat{C}_b^t \delta \vec{\omega}_{ib}^b + \delta \vec{\psi}_{tb}^t \times \vec{\omega}_{ie}^t
 \end{aligned}$$

$$\delta \dot{\vec{\psi}}_{tb}^t = \hat{C}_b^t \delta \vec{\omega}_{ib}^b - \vec{\Omega}_{ie}^t \delta \vec{\psi}_{tb}^t \quad (4)$$

$$\dot{\vec{v}}_{tb}^t = C_b^t \vec{f}_{ib}^b + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^t \quad (5)$$

$$\begin{aligned} \hat{\dot{\vec{v}}}_{tb}^t &= \hat{C}_b^t \hat{\vec{f}}_{ib}^b + \hat{\vec{g}}_b^t - 2\Omega_{ie}^t \hat{\vec{v}}_{tb}^t \\ &= (\mathcal{I} - [\delta\vec{\psi}_{tb}^t \times]) C_b^t (\vec{f}_{ib}^b - \delta\vec{f}_{ib}^b) + \hat{\vec{g}}_b^t - 2\Omega_{ie}^t \hat{\vec{v}}_{tb}^t \end{aligned} \quad (6)$$

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$$\begin{aligned} \delta\dot{\vec{v}}_{tb}^t &= \dot{\vec{v}}_{tb}^t - \hat{\dot{\vec{v}}}_{tb}^t = [\delta\vec{\psi}_{tb}^t \times] C_b^t \vec{f}_{ib}^b + \hat{C}_b^t \delta\vec{f}_{ib}^b + \delta\vec{g}_b^t - 2\Omega_{ie}^t \delta\vec{v}_{tb}^t \\ &= [\delta\vec{\psi}_{tb}^t \times] \hat{C}_b^t \hat{\vec{f}}_{ib}^b + \hat{C}_b^t \delta\vec{f}_{ib}^b + \delta\vec{g}_b^t - 2\Omega_{ie}^t \delta\vec{v}_{tb}^t \end{aligned}$$

$$\dot{\vec{v}}_{tb}^t = C_b^t \vec{f}_{ib}^b + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^t \quad (5)$$

$$\begin{aligned} \hat{\dot{\vec{v}}}_{tb}^t &= \hat{C}_b^t \hat{\vec{f}}_{ib}^b + \hat{\vec{g}}_b^t - 2\Omega_{ie}^t \hat{\vec{v}}_{tb}^t \\ &= (\mathcal{I} - [\delta\vec{\psi}_{tb}^t \times]) C_b^t (\vec{f}_{ib}^b - \delta\vec{f}_{ib}^b) + \hat{\vec{g}}_b^t - 2\Omega_{ie}^t \hat{\vec{v}}_{tb}^t \end{aligned} \quad (6)$$

$$\begin{aligned} \delta\dot{\vec{v}}_{tb}^t &= \dot{\vec{v}}_{tb}^t - \hat{\dot{\vec{v}}}_{tb}^t = [\delta\vec{\psi}_{tb}^t \times] C_b^t \vec{f}_{ib}^b + \hat{C}_b^t \delta\vec{f}_{ib}^b + \delta\vec{g}_b^t - 2\Omega_{ie}^t \delta\vec{v}_{tb}^t \\ &= [\delta\vec{\psi}_{tb}^t \times] \hat{C}_b^t \hat{\vec{f}}_{ib}^b + \hat{C}_b^t \delta\vec{f}_{ib}^b + \delta\vec{g}_b^t - 2\Omega_{ie}^t \delta\vec{v}_{tb}^t \end{aligned}$$

$$\delta\dot{\vec{v}}_{tb}^t = -[\hat{C}_b^t \hat{\vec{f}}_{ib}^b \times] \delta\vec{\psi}_{tb}^t + \hat{C}_b^t \delta\vec{f}_{ib}^b + \delta\vec{g}_b^t - 2\Omega_{ie}^t \delta\vec{v}_{tb}^t \quad (7)$$

$$\delta \vec{g}_b^t \approx \hat{C}_e^t \frac{2g_0(\hat{L}_b)}{r_{eS}^e(\hat{L}_b)} \frac{\hat{r}_{eb}^e}{|\hat{r}_{eb}^e|^2} (\hat{r}_{eb}^e)^T \hat{C}_t^e \delta \vec{r}_{tb}^t \quad (8)$$

$$\dot{\vec{r}}_{tb}^t = \vec{v}_{tb}^t \quad (9)$$

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$$\delta \dot{\vec{r}}_{tb}^t = \delta \vec{v}_{tb}^t \quad (10)$$

$$\begin{pmatrix} \delta\dot{\psi}_{tb}^t \\ \delta\dot{\mathbf{v}}_{tb}^t \\ \delta\dot{\mathbf{r}}_{tb}^t \end{pmatrix} = \begin{bmatrix} -\Omega_{ie}^t & \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ -[\hat{\mathbf{C}}_b^t \hat{\mathbf{f}}_{ib}^b \times] & -2\Omega_{ie}^t & \hat{\mathbf{C}}_e^t \frac{2g_0(\hat{\mathbf{L}}_b)}{r_{eS}^e(\hat{\mathbf{L}}_b)} \frac{\hat{\mathbf{r}}_{eb}^e}{|\hat{\mathbf{r}}_{eb}^e|^2} (\hat{\mathbf{r}}_{eb}^e)^T \hat{\mathbf{C}}_t^e \\ \mathbf{0}_{3 \times 3} & \mathbf{I}_{3 \times 3} & \mathbf{0}_{3 \times 3} \end{bmatrix} \begin{pmatrix} \delta\psi_{tb}^t \\ \delta\mathbf{v}_{tb}^t \\ \delta\mathbf{r}_{tb}^t \end{pmatrix} + \begin{bmatrix} 0 & \hat{\mathbf{C}}_b^t \\ \hat{\mathbf{C}}_b^t & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \delta\vec{f}_{ib}^b \\ \delta\vec{\omega}_{ib}^b \end{pmatrix} \quad (11)$$

- Truth value

 \vec{x}

- Measured value

 $\tilde{\vec{x}}$

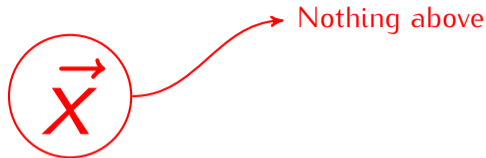
- Estimated or computed value

 $\hat{\vec{x}}$

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value
- Measured value
- Estimated or computed value
- Error

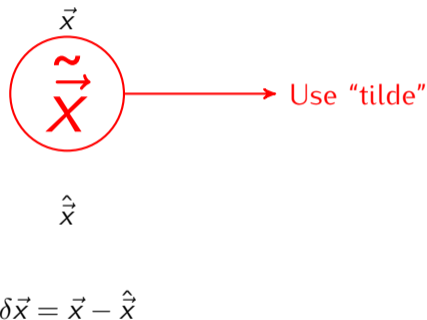


\vec{x}

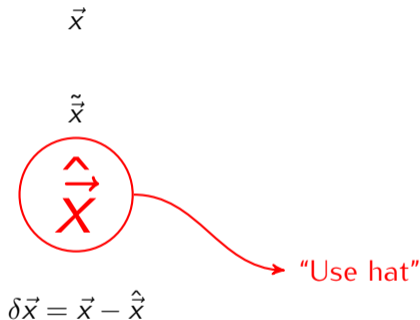
$\hat{\vec{x}}$

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value
- Measured value
- Estimated or computed value
- Error



- Truth value
- Measured value
- **Estimated or computed value**
- Error



- Truth value

\vec{x}

- Measured value

$\tilde{\vec{x}}$

- Estimated or computed value

$\hat{\vec{x}}$

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$

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Let's assume we have an estimate of \vec{x} , i.e., $\hat{\vec{x}}$ such that $\vec{x} = \hat{\vec{x}} + \delta\vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \quad (12)$$

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$

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$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \quad (12)$$

Using Taylor series expansion

$$\begin{aligned} f(\hat{\vec{x}} + \delta\vec{x}, t) = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} &= f(\hat{\vec{x}}, t) + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T \\ &\approx \dot{\hat{\vec{x}}} + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \end{aligned}$$

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$

Let's assume we have an estimate of \vec{x} , i.e., $\hat{\vec{x}}$ such that $\vec{x} = \hat{\vec{x}} + \delta\vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \quad (12)$$

Using Taylor series expansion

$$\begin{aligned} f(\hat{\vec{x}} + \delta\vec{x}, t) &= \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = \underbrace{f(\hat{\vec{x}}, t)}_{=} + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T \\ &\approx \underbrace{\dot{\hat{\vec{x}}}}_{=} + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \end{aligned}$$

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$

Let's assume we have an estimate of \vec{x} , i.e., $\hat{\vec{x}}$ such that $\vec{x} = \hat{\vec{x}} + \delta\vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \quad (12)$$

Using Taylor series expansion

$$\begin{aligned} f(\hat{\vec{x}} + \delta\vec{x}, t) = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} &= f(\hat{\vec{x}}, t) + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T \\ &\approx \dot{\hat{\vec{x}}} + \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \end{aligned}$$

$$\Rightarrow \delta\dot{\vec{x}} \approx \left. \frac{\partial f(\vec{x}, t)}{\partial \vec{x}} \right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \quad (13)$$

Initially the accelerometer and gyroscope measurements, $\tilde{\vec{f}}_{ib}^b$ and $\tilde{\vec{\omega}}_{ib}^b$, respectively, will be modeled as

$$\tilde{\vec{f}}_{ib}^b = \vec{f}_{ib}^b + \Delta\vec{f}_{ib}^b \quad (14)$$

$$\tilde{\vec{\omega}}_{ib}^b = \vec{\omega}_{ib}^b + \Delta\vec{\omega}_{ib}^b \quad (15)$$

where \vec{f}_{ib}^b and $\vec{\omega}_{ib}^b$ are the specific force and angular rates, respectively; and $\Delta\vec{f}_{ib}^b$ and $\Delta\vec{\omega}_{ib}^b$ represents the errors. In later lectures we will discuss more detailed description of these errors.

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$$\tilde{\vec{f}}_{ib}^b = \vec{f}_{ib}^b + \Delta\vec{f}_{ib}^b \quad \text{these terms may} \quad (14)$$

$$\tilde{\vec{\omega}}_{ib}^b = \vec{\omega}_{ib}^b + \Delta\vec{\omega}_{ib}^b \quad \text{be expanded further} \quad (15)$$

where \vec{f}_{ib}^b and $\vec{\omega}_{ib}^b$ are the specific force and angular rates, respectively; and $\Delta\vec{f}_{ib}^b$ and $\Delta\vec{\omega}_{ib}^b$ represents the errors. In later lectures we will discuss more detailed description of these errors.

Accelerometers

$$\tilde{\vec{f}}_{ib}^b = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^b + \vec{n}l_a + \vec{w}_a \quad (16)$$

Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^b = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^b + G_g\vec{f}_{ib}^b + \vec{w}_g \quad (17)$$

Accelerometers

$$\tilde{\vec{f}}_{ib}^b = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^b + \vec{n}l_a + \vec{w}_a \quad (16)$$

Biases

Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^b = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^b + G_g\vec{f}_{ib}^b + \vec{w}_g \quad (17)$$

Accelerometers

$$\tilde{\vec{f}}_{ib}^b = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^b + \vec{n}l_a + \vec{w}_a \quad (16)$$

Misalignment and SF Errors



Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^b = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^b + G_g\vec{f}_{ib}^b + \vec{w}_g \quad (17)$$

Accelerometers

$$\tilde{\vec{f}}_{ib}^b = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^b + \vec{nl}_a + \vec{w}_a \quad (16)$$

Non-linearity

Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^b = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^b + G_g\vec{f}_{ib}^b + \vec{w}_g \quad (17)$$

Accelerometers

$$\tilde{\vec{f}}_{ib}^b = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^b + \vec{n}l_a + \vec{w}_a \quad (16)$$

G-Sensitivity

Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^b = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^b + \underbrace{G_g \vec{f}_{ib}^b}_{\text{G-Sensitivity}} + \vec{w}_g \quad (17)$$

Accelerometers

$$\tilde{\vec{f}}_{ib}^b = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^b + \vec{n}l_a + \vec{w}_a \quad (16)$$

Noise

Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^b = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^b + G_g\vec{f}_{ib}^b + \vec{w}_g \quad (17)$$

- Position error

$$\delta \vec{r}_{\beta b}^{\gamma} = \vec{r}_{\beta b}^{\gamma} - \hat{\vec{r}}_{\beta b}^{\gamma} \quad (18)$$

- Velocity error

$$\delta \vec{v}_{\beta b}^{\gamma} = \vec{v}_{\beta b}^{\gamma} - \hat{\vec{v}}_{\beta b}^{\gamma} \quad (19)$$

- Specific force errors

$$\delta \vec{f}_{ib}^b = \vec{f}_{ib}^b - \hat{\vec{f}}_{ib}^b \quad (20)$$

$$\Delta_e \vec{f}_{ib}^b = \Delta \vec{f}_{ib}^b - \Delta \hat{\vec{f}}_{ib}^b = -\delta \vec{f}_{ib}^b \quad (21)$$

- Angular rate errors

$$\delta \vec{\omega}_{ib}^b = \vec{\omega}_{ib}^b - \hat{\vec{\omega}}_{ib}^b \quad (22)$$

$$\Delta_e \vec{\omega}_{ib}^b = \Delta \vec{\omega}_{ib}^b - \Delta \hat{\vec{\omega}}_{ib}^b = -\delta \vec{\omega}_{ib}^b \quad (23)$$

Define

$$\delta C_b^\gamma = C_b^\gamma \hat{C}_\gamma^b = e^{[\delta \vec{\psi}_{\gamma b}^\gamma \times]} \approx \mathcal{I} + [\delta \vec{\psi}_{\gamma b}^\gamma \times] \quad (24)$$

This is the error in attitude resulting from errors in estimating the angular rates.

The attitude error is a multiplicative small angle transformation from the actual frame to the computed frame

$$\hat{C}_b^\gamma = (\mathcal{I} - [\delta\vec{\psi}_{\gamma b}^\gamma \times]) C_b^\gamma \quad (25)$$

Similarly,

$$C_b^\gamma = (\mathcal{I} + [\delta\vec{\psi}_{\gamma b}^\gamma \times]) \hat{C}_b^\gamma \quad (26)$$

Similarly we can attempt to estimate the specific force and angular rate by applying correction based on our estimate of the error.

$$\hat{\vec{f}}_{ib}^b = \tilde{\vec{f}}_{ib}^b - \Delta \hat{\vec{f}}_{ib}^b \quad (27)$$

$$\hat{\vec{\omega}}_{ib}^b = \tilde{\vec{\omega}}_{ib}^b - \Delta \hat{\vec{\omega}}_{ib}^b \quad (28)$$

where $\hat{\vec{f}}_{ib}^b$ and $\hat{\vec{\omega}}_{ib}^b$ are the accelerometer and gyroscope estimated calibration values, respectively.