EE 570: Location and Navigation

Error Mechanization (ECEF)

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ECEF Attitude Error

$$\dot{C}_b^e = C_b^e \Omega_{eb}^b \; = C_b^e (\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt}\left[\left(\mathcal{I} + [\delta \vec{\psi}_{eb}^e \times] \right) \hat{C}_b^e \right] =$$

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$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\,\hat{C}_b^e \Omega_{eb}^b = [\delta\dot{\vec{\psi}}_{eb}^e \times]\,\hat{C}_b^e + (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\,\dot{\hat{C}}_b^e =$$

$$\dot{C}_b^e = C_b^e \Omega_{eb}^b = C_b^e(\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e \Omega_{eb}^b = [\delta\dot{\vec{\psi}}_{eb}^e \times]\hat{C}_b^e + (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\dot{\hat{C}}_b^e =$$
$$\approx (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e(\hat{\Omega}_{eb}^b + \delta\Omega_{ib}^b - \delta\Omega_{ie}^b)$$

$$\dot{C}_b^e = C_b^e \Omega_{eb}^b \ = C_b^e(\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\,\hat{C}_b^e\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\,\hat{C}_b^e \Omega_{eb}^b = [\delta\dot{\vec{\psi}}_{eb}^e \times]\,\hat{C}_b^e + (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\,\dot{\hat{C}}_b^e =$$

$$\approx (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\,\hat{C}_b^e(\hat{\Omega}_{eb}^b + \delta\Omega_{ib}^b - \delta\Omega_{ie}^b)$$

$$\approx (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\,\hat{C}_b^e\hat{\Omega}_{eb}^b + \hat{C}_b^e(\delta\Omega_{ib}^b - \delta\Omega_{ie}^b)$$

$$\because [\delta\vec{\psi}_{eb}^e \times]\delta\Omega_{eb}^b \approx 0$$

$$\dot{C}_b^e = C_b^e \Omega_{eb}^b = C_b^e(\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e \Omega_{eb}^b = [\delta\dot{\vec{\psi}}_{eb}^e \times]\hat{C}_b^e + (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\dot{\hat{C}}_b^e =$$

$$\approx (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e(\hat{\Omega}_{eb}^b + \delta\Omega_{ib}^b - \delta\Omega_{ie}^b)$$

$$=$$

$$\approx (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e\hat{\Omega}_{eb}^b + \hat{C}_b^e(\delta\Omega_{ib}^b - \delta\Omega_{ie}^b)$$

ECEF Attitude Error

$$\dot{C}_b^e = C_b^e \Omega_{eb}^b = C_b^e (\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt}\left[\left(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]\right) \hat{C}_b^e \right] =$$

$$\left(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]\right) \hat{C}_b^e \Omega_{eb}^b = [\delta\dot{\vec{\psi}}_{eb}^e \times] \hat{C}_b^e + \left(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]\right) \dot{\hat{C}}_b^e =$$

$$\approx \left(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]\right) \hat{C}_b^e (\hat{\Omega}_{eb}^b + \delta\Omega_{ib}^b - \delta\Omega_{ie}^b)$$

$$\approx \left(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times]\right) \hat{C}_b^e \hat{\Omega}_{eb}^b + \hat{C}_b^e (\delta\Omega_{ib}^b - \delta\Omega_{ie}^b)$$

$$[\delta\dot{\vec{\psi}}_{eb}^e \times] = \hat{C}_b^e (\delta\Omega_{ib}^b - \delta\Omega_{ie}^b) \hat{C}_e^b = [\hat{C}_b^e (\delta\vec{\omega}_{ib}^b - \delta\vec{\omega}_{ie}^b) \times] \tag{1}$$

ECEF Attitude Error

$$\dot{C}_b^e = C_b^e \Omega_{eb}^b = C_b^e(\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e \Omega_{eb}^b = [\delta\dot{\vec{\psi}}_{eb}^e \times]\hat{C}_b^e + (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\dot{\hat{C}}_b^e =$$
$$\approx (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e(\hat{\Omega}_{eb}^b + \delta\Omega_{ib}^b - \delta\Omega_{ie}^b)$$
$$\approx (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e\hat{\Omega}_{eb}^b + \hat{C}_b^e(\delta\Omega_{ib}^b - \delta\Omega_{ie}^b)$$

$$[\delta\dot{\vec{\psi}}_{eb}^e \times] = \hat{C}_b^e(\delta\Omega_{ib}^b - \delta\Omega_{ie}^b)\hat{C}_e^b = [\hat{C}_b^e(\delta\vec{\omega}_{ib}^b - \delta\vec{\omega}_{ie}^b)\times] \tag{1}$$

$$\delta\dot{\vec{\psi}}_{eb}^e = \hat{C}_b^e(\delta\omega_{ib}^b - \delta\vec{\omega}_{ie}^b) \tag{2}$$

$$
\begin{aligned}
\delta\dot{\vec{\psi}}^{e}_{eb} &= \hat{C}^{e}_{b}(\delta\omega^{b}_{ib} - \delta\vec{\omega}^{b}_{ie}) \\
&= \hat{C}^{e}_{b}\delta\omega^{b}_{ib} - \hat{C}^{e}_{b}(\vec{\omega}^{b}_{ie} - \hat{\vec{\omega}}^{b}_{ie}) \\
&= \hat{C}^{e}_{b}\delta\omega^{b}_{ib} - (\hat{C}^{e}_{b}C^{b}_{e} - \mathcal{I})\vec{\omega}^{e}_{ie} \\
&= \hat{C}^{e}_{b}\delta\omega^{b}_{ib} + \delta\vec{\psi}^{e}_{eb} \times \vec{\omega}^{e}_{ie}
\end{aligned}
$$

$$
\begin{aligned}
\delta\dot{\vec{\psi}}^{e}_{eb} &= \hat{C}^e_b(\delta\omega^b_{ib} - \delta\vec{\omega}^b_{ie}) \\
&= \hat{C}^e_b\delta\omega^b_{ib} - \hat{C}^e_b(\vec{\omega}^b_{ie} - \hat{\vec{\omega}}^b_{ie}) \\
&= \hat{C}^e_b\delta\omega^b_{ib} - (\hat{C}^e_b C^b_e - \mathcal{I})\vec{\omega}^e_{ie} \\
&= \hat{C}^e_b\delta\omega^b_{ib} + \delta\vec{\psi}^e_{eb} \times \vec{\omega}^e_{ie}
\end{aligned}
$$

$$
\delta\dot{\vec{\psi}}^{e}_{eb} = \hat{C}^e_b\delta\omega^b_{ib} - \vec{\Omega}^e_{ie}\delta\vec{\psi}^e_{eb} \tag{3}
$$

$$\dot{\vec{v}}_{eb}^{e} = C_b^e \vec{f}_{ib}^{b} + \vec{g}_b^e - 2\Omega_{ie}^e \vec{v}_{eb}^{e} \tag{4}$$

$$\begin{aligned} \dot{\hat{\vec{v}}}_{eb}^{e} &= \hat{C}_b^e \hat{\vec{f}}_{ib}^{b} + \hat{\vec{g}}_b^e - 2\Omega_{ie}^e \hat{\vec{v}}_{eb}^{e} \\ &= \left(\mathcal{I} - [\delta\vec{\psi}_{eb}^{e}\times]\right) C_b^e \left(\vec{f}_{ib}^{b} - \delta\vec{f}_{ib}^{b}\right) + \hat{\vec{g}}_b^e - 2\Omega_{ie}^e \hat{\vec{v}}_{eb}^{e} \end{aligned} \tag{5}$$

Velocity

$$\dot{\vec{v}}^e_{eb} = C^e_b \vec{f}^b_{ib} + \vec{g}^e_b - 2\Omega^e_{ie} \vec{v}^e_{eb} \tag{4}$$

$$\begin{aligned}
\dot{\hat{\vec{v}}}^e_{eb} &= \hat{C}^e_b \hat{\vec{f}}^b_{ib} + \hat{\vec{g}}^e_b - 2\Omega^e_{ie} \hat{\vec{v}}^e_{eb} \\
&= (\mathcal{I} - [\delta\vec{\psi}^e_{eb}\times]) C^e_b (\vec{f}^b_{ib} - \delta\vec{f}^b_{ib}) + \hat{\vec{g}}^e_b - 2\Omega^e_{ie} \hat{\vec{v}}^e_{eb}
\end{aligned} \tag{5}$$

$$\begin{aligned}
\delta\dot{\vec{v}}^e_{eb} = \dot{\vec{v}}^e_{eb} - \dot{\hat{\vec{v}}}^e_{eb} &= [\delta\vec{\psi}^e_{eb}\times] C^e_b \vec{f}^b_{ib} + \hat{C}^e_b \delta\vec{f}^b_{ib} + \delta\vec{g}^e_b - 2\Omega^e_{ie} \delta\vec{v}^e_{eb} \\
&= [\delta\vec{\psi}^e_{eb}\times] \hat{C}^e_b \hat{\vec{f}}^b_{ib} + \hat{C}^e_b \delta\vec{f}^b_{ib} + \delta\vec{g}^e_b - 2\Omega^e_{ie} \delta\vec{v}^e_{eb}
\end{aligned}$$

Velocity

$$\dot{\vec{v}}^e_{eb} = C^e_b \vec{f}^b_{ib} + \vec{g}^e_b - 2\Omega^e_{ie}\vec{v}^e_{eb} \tag{4}$$

$$\begin{aligned}\dot{\hat{\vec{v}}}^e_{eb} &= \hat{C}^e_b \hat{\vec{f}}^b_{ib} + \hat{\vec{g}}^e_b - 2\Omega^e_{ie}\hat{\vec{v}}^e_{eb} \\ &= (\mathcal{I} - [\delta\vec{\psi}^e_{eb}\times])C^e_b(\vec{f}^b_{ib} - \delta\vec{f}^b_{ib}) + \hat{\vec{g}}^e_b - 2\Omega^e_{ie}\hat{\vec{v}}^e_{eb}\end{aligned} \tag{5}$$

$$\begin{aligned}\delta\dot{\vec{v}}^e_{eb} = \dot{\vec{v}}^e_{eb} - \dot{\hat{\vec{v}}}^e_{eb} &= [\delta\vec{\psi}^e_{eb}\times]C^e_b\vec{f}^b_{ib} + \hat{C}^e_b\delta\vec{f}^b_{ib} + \delta\vec{g}^e_b - 2\Omega^e_{ie}\delta\vec{v}^e_{eb} \\ &= [\delta\vec{\psi}^e_{eb}\times]\hat{C}^e_b\hat{\vec{f}}^b_{ib} + \hat{C}^e_b\delta\vec{f}^b_{ib} + \delta\vec{g}^e_b - 2\Omega^e_{ie}\delta\vec{v}^e_{eb}\end{aligned}$$

$$\delta\dot{\vec{v}}^e_{eb} = -[\hat{C}^e_b\hat{\vec{f}}^b_{ib}\times]\delta\vec{\psi}^e_{eb} + \hat{C}^e_b\delta\vec{f}^b_{ib} + \delta\vec{g}^e_b - 2\Omega^e_{ie}\hat{\vec{v}}^e_{eb} \tag{6}$$

Gravity Error

$$\delta \vec{g}_b^e \approx \frac{2g_0(\hat{L}_b)}{r_{eS}^e(\hat{L}_b)} \frac{\hat{\vec{r}}_{eb}^e}{|\hat{\vec{r}}_{eb}^e|^2} (\hat{\vec{r}}_{eb}^e)^T \delta \vec{r}_{eb}^e \tag{7}$$

Position

$$\dot{\vec{r}}_{eb}^{e} = \vec{v}_{eb}^{e} \tag{8}$$

$$\dot{\vec{r}}^{e}_{eb} = \vec{v}^{e}_{eb} \tag{8}$$

$$\delta\dot{\vec{r}}^{e}_{eb} = \delta\vec{v}^{e}_{eb} \tag{9}$$

Summary - in terms of $\delta\vec{f}^b_{ib}, \delta\vec{\omega}^b_{ib}$

$$\begin{pmatrix} \delta\dot{\vec{\psi}}^e_{eb} \\ \delta\dot{\vec{v}}^e_{eb} \\ \delta\dot{\vec{r}}^e_{eb} \end{pmatrix} = \begin{bmatrix} -\Omega^e_{ie} & 0_{3\times3} & 0_{3\times3} \\ -[\hat{C}^e_b \hat{\vec{f}}^b_{ib}\times] & -2\Omega^e_{ie} & \frac{2g_0(\hat{L}_b)}{r^e_{eS}(\hat{L}_b)} \frac{\hat{\vec{r}}^e_{eb}}{|\hat{\vec{r}}^e_{eb}|^2}\left(\hat{\vec{r}}^e_{eb}\right)^T \\ 0_{3\times3} & \mathcal{I}_{3\times3} & 0_{3\times3} \end{bmatrix} \begin{pmatrix} \delta\vec{\psi}^e_{eb} \\ \delta\vec{v}^e_{eb} \\ \delta\vec{r}^e_{eb} \end{pmatrix} + \begin{bmatrix} 0 & \hat{C}^e_b \\ \hat{C}^e_b & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \delta\vec{f}^b_{ib} \\ \delta\vec{\omega}^b_{ib} \end{pmatrix} \tag{10}$$

- Truth value

$$\vec{x}$$

- Measured value

$$\tilde{\vec{x}}$$

- Estimated or computed value

$$\hat{\vec{x}}$$

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value

$$\vec{x}$$

Nothing above

- Measured value

$$\tilde{\vec{x}}$$

- Estimated or computed value

$$\hat{\vec{x}}$$

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value

$$\vec{x}$$

- Measured value

$$\tilde{\vec{x}}$$

Use "tilde"

- Estimated or computed value

$$\hat{\vec{x}}$$

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value

$$\vec{x}$$

- Measured value

$$\tilde{\vec{x}}$$

- Estimated or computed value

$$\hat{\vec{x}}$$

"Use hat"

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value

$$\vec{x}$$

- Measured value

$$\tilde{\vec{x}}$$

- Estimated or computed value

$$\hat{\vec{x}}$$

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

Linearization using Taylor Series Expansion

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$

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Let's assume we have an estimate of $\vec{x}$, i.e., $\hat{\vec{x}}$ such that $\vec{x} = \hat{\vec{x}} + \delta\vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \tag{11}$$

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Using Taylor series expansion

$$\begin{aligned} f(\hat{\vec{x}} + \delta\vec{x}, t) = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = \; & f(\hat{\vec{x}}, t) + \left.\frac{\partial f(\vec{x}, t)}{\partial\vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T \\ \approx \; & \dot{\hat{\vec{x}}} + \left.\frac{\partial f(\vec{x}, t)}{\partial\vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \end{aligned}$$

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Using Taylor series expansion

$$f(\hat{\vec{x}} + \delta\vec{x}, t) = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}}, t) + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T$$
$$\approx \dot{\hat{\vec{x}}} + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x}$$

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Let's assume we have an estimate of $\vec{x}$, i.e., $\hat{\vec{x}}$ such that $\vec{x} = \hat{\vec{x}} + \delta\vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \tag{11}$$

Using Taylor series expansion

$$\begin{aligned} f(\hat{\vec{x}} + \delta\vec{x}, t) = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = \ & f(\hat{\vec{x}}, t) + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T \\ \approx \ & \dot{\hat{\vec{x}}} + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \end{aligned}$$

$$\Rightarrow \delta\dot{\vec{x}} \approx \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \tag{12}$$

Initially the accelerometer and gyroscope measurements, $\tilde{\vec{f}}_{ib}^{b}$ and $\tilde{\vec{\omega}}_{ib}^{b}$, respectively, will be modeled as

$$\tilde{\vec{f}}_{ib}^{b} = \vec{f}_{ib}^{b} + \Delta\vec{f}_{ib}^{b} \tag{13}$$

$$\tilde{\vec{\omega}}_{ib}^{b} = \vec{\omega}_{ib}^{b} + \Delta\vec{\omega}_{ib}^{b} \tag{14}$$

where $\vec{f}_{ib}^{b}$ and $\vec{\omega}_{ib}^{b}$ are the specific force and angular rates, respectively; and $\Delta\vec{f}_{ib}^{b}$ and $\Delta\vec{\omega}_{ib}^{b}$ represents the errors. In later lectures we will discuss more detailed description of these errors.

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$$\tilde{\vec{\omega}}_{ib}^{b} = \vec{\omega}_{ib}^{b} + \Delta\vec{\omega}_{ib}^{b} \tag{14}$$

these terms may be expanded further

where $\vec{f}_{ib}^{b}$ and $\vec{\omega}_{ib}^{b}$ are the specific force and angular rates, respectively; and $\Delta\vec{f}_{ib}^{b}$ and $\Delta\vec{\omega}_{ib}^{b}$ represents the errors. In later lectures we will discuss more detailed description of these errors.

Accelerometers

$$\tilde{\vec{f}}_{ib}^{b} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^{b} + \vec{nl}_a + \vec{w}_a \tag{15}$$

Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^{b} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^{b} + G_g \vec{f}_{ib}^{b} + \vec{w}_g \tag{16}$$

Error Modeling Example

Accelerometers

$$\tilde{\vec{f}}^b_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}^b_{ib} + \vec{nl}_a + \vec{w}_a \tag{15}$$

Biases

Gyroscopes

$$\tilde{\vec{\omega}}^b_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}^b_{ib} + G_g\vec{f}^b_{ib} + \vec{w}_g \tag{16}$$

Accelerometers

$$\tilde{\vec{f}}_{ib}^{b} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^{b} + \vec{nl}_a + \vec{w}_a \tag{15}$$

Misalignment and SF Errors

Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^{b} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^{b} + G_g\vec{f}_{ib}^{b} + \vec{w}_g \tag{16}$$

Error Modeling Example

Accelerometers

$$\tilde{\vec{f}}_{ib}^{b} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^{b} + \vec{nl}_a + \vec{w}_a \quad (15)$$

Non-linearity

Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^{b} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^{b} + G_g \vec{f}_{ib}^{b} + \vec{w}_g \quad (16)$$

Accelerometers

$$\tilde{\vec{f}}_{ib}^{b} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^{b} + \vec{nl}_a + \vec{w}_a \tag{15}$$

G-Sensitivity

Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^{b} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^{b} + G_g\vec{f}_{ib}^{b} + \vec{w}_g \tag{16}$$

Accelerometers

$$\tilde{\vec{f}}_{ib}^{b} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^{b} + \vec{nl}_a + \vec{w}_a \tag{15}$$

Noise

Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^{b} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^{b} + G_g \vec{f}_{ib}^{b} + \vec{w}_g \tag{16}$$

- Position error

$$\delta\vec{r}^{\gamma}_{\beta b} = \vec{r}^{\gamma}_{\beta b} - \hat{\vec{r}}^{\gamma}_{\beta b} \quad (17)$$

- Velocity error

$$\delta\vec{v}^{\gamma}_{\beta b} = \vec{v}^{\gamma}_{\beta b} - \hat{\vec{v}}^{\gamma}_{\beta b} \quad (18)$$

- Specific force errors

$$\delta\vec{f}^{b}_{ib} = \vec{f}^{b}_{ib} - \hat{\vec{f}}^{b}_{ib} \quad (19)$$

$$\Delta_e\vec{f}^{b}_{ib} = \Delta\vec{f}^{b}_{ib} - \Delta\hat{\vec{f}}^{b}_{ib} = -\delta\vec{f}^{b}_{ib} \quad (20)$$

- Angular rate errors

$$\delta\vec{\omega}^{b}_{ib} = \vec{\omega}^{b}_{ib} - \hat{\vec{\omega}}^{b}_{ib} \quad (21)$$

$$\Delta_e\vec{\omega}^{b}_{ib} = \Delta\vec{\omega}^{b}_{ib} - \Delta\hat{\vec{\omega}}^{b}_{ib} = -\delta\vec{\omega}^{b}_{ib} \quad (22)$$

Define

$$\delta C_b^\gamma = C_b^\gamma \hat{C}_\gamma^b = e^{[\delta\vec{\psi}_{\gamma b}^\gamma \times]} \approx \mathcal{I} + [\delta\vec{\psi}_{\gamma b}^\gamma \times] \tag{23}$$

This is the error in attitude resulting from errors in estimating the angular rates.

The attitude error is a multiplicative small angle transformation from the actual frame to the computed frame

$$\hat{C}_b^\gamma = (\mathcal{I} - [\delta\vec{\psi}_{\gamma b}^\gamma \times]) C_b^\gamma \tag{24}$$

Similarly,

$$C_b^\gamma = (\mathcal{I} + [\delta\vec{\psi}_{\gamma b}^\gamma \times]) \hat{C}_b^\gamma \tag{25}$$

Similarly we can attempt to estimate the specific force and angular rate by applying correction based on our estimate of the error.

$$\hat{\vec{f}}^{b}_{ib} = \tilde{\vec{f}}^{b}_{ib} - \Delta\hat{\vec{f}}^{b}_{ib} \tag{26}$$

$$\hat{\vec{\omega}}^{b}_{ib} = \tilde{\vec{\omega}}^{b}_{ib} - \Delta\hat{\vec{\omega}}^{b}_{ib} \tag{27}$$

where $\hat{\vec{f}}^{b}_{ib}$ and $\hat{\vec{\omega}}^{b}_{ib}$ are the accelerometer and gyroscope estimated calibration values, respectively.