# Lecture
# Error Mechanization (ECI)
## EE 570: Location and Navigation

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## 1 Overview

### Mechanization

We have already derived the kinematic models in several frames. These models may be written in the form

$$\dot{\vec{x}} = \mathbf{f}(\vec{x}, \vec{u}) \tag{1}$$

where $\mathbf{f}$ is possibly non-linear. .2

### In Reality

Due to errors in the measurements we estimate $\vec{x}$ by integrating

$$\dot{\hat{\vec{x}}} = \mathbf{f}(\hat{\vec{x}}, \hat{\vec{u}}) \tag{2}$$

where $\hat{\vec{u}}$ is the measurement vector from the sensors after applying calibration corrections.

*If somehow we can model and possibly measure the error in the state we can then subtract it from the estimate to obtain an accurate position, velocity and attitude. We may also want to linearize the problem so that linear estimation approaches could be used.* .3

### Gyro and Accel Measurement Errors

All accelerometers and gyroscopes suffer from

- Biases
- Scale factor
- Cross-coupling
- Random noise .4

### Components of Measurement Errors

- *Fixed Errors*: deterministic and are present all the time, hence can be removed using calibration.
- *Temperature Dependent*: variations dependent on temperature and also may be modeled and characterized during calibration.
- *Run-to-run*: changes in the sensor error every time the sensor is run and is random in nature.
- *In-run*: random variations as the sensor is running.

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# 2 Preliminaries

## 2.1 Basic Definitions

### Notation Used

- *Truth value*

$$\vec{x}$$

- *Measured value*

$$\tilde{\vec{x}}$$

- *Estimated or computed value*

$$\hat{\vec{x}}$$

- *Error*

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

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## 2.2 Linearization

### Linearization using Taylor Series Expansion

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$

Let's assume we have an estimate of $\vec{x}$, i.e., $\hat{\vec{x}}$ such that $\vec{x} = \hat{\vec{x}} + \delta\vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \tag{3}$$

Using Taylor series expansion

$$\begin{aligned} f(\hat{\vec{x}} + \delta\vec{x}, t) = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = \; & f(\hat{\vec{x}}, t) \; + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T \\ \approx \; & \dot{\hat{\vec{x}}} \; + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \end{aligned}$$

$$\Rightarrow \delta\dot{\vec{x}} \approx \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \tag{4}$$

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# 3 Inertial Measurements

### Actual Measurements

Initially the accelerometer and gyroscope measurements, $\tilde{\vec{f}}{}^{b}_{ib}$ and $\tilde{\vec{\omega}}{}^{b}_{ib}$, respectively, will be modeled as

$$\tilde{\vec{f}}{}^{b}_{ib} = \vec{f}{}^{b}_{ib} + \Delta\vec{f}{}^{b}_{ib} \tag{5}$$

$$\tilde{\vec{\omega}}{}^{b}_{ib} = \vec{\omega}{}^{b}_{ib} + \Delta\vec{\omega}{}^{b}_{ib} \tag{6}$$

where $\vec{f}{}^{b}_{ib}$ and $\vec{\omega}{}^{b}_{ib}$ are the specific force and angular rates, respectively; and $\Delta\vec{f}{}^{b}_{ib}$ and $\Delta\vec{\omega}{}^{b}_{ib}$ represents the errors. In later lectures we will discuss more detailed description of these errors. .8

### Error Modeling Example

Accelerometers

$$\tilde{\vec{f}}^{\,b}_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}^{\,b}_{ib} + \vec{nl}_a + \vec{w}_a \tag{7}$$

Gyroscopes

$$\tilde{\vec{\omega}}^{\,b}_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}^{\,b}_{ib} + G_g\vec{f}^{\,b}_{ib} + \vec{w}_g \tag{8}$$

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### State Error Vector

Define the error state vector as

$$\delta\vec{x}^{\,\gamma}_{INS} = \begin{pmatrix} \delta\vec{\psi}^{\,\gamma}_{\gamma b} \\ \delta\vec{v}^{\,\gamma}_{\beta b} \\ \delta\vec{r}^{\,\gamma}_{\beta b} \end{pmatrix}, \quad \gamma, \beta \in i, e, n \tag{9}$$

Think of $\delta\vec{x}$ as the truth minus the estimate, i.e.,

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}} \tag{10}$$

*The subtraction doesn't apply to the attitude component of the vector and needs to be treated differently*

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### Pos, Vel, Force and Angular Rate Errors

- Position error

$$\delta\vec{r}^{\,\gamma}_{\beta b} = \vec{r}^{\,\gamma}_{\beta b} - \hat{\vec{r}}^{\,\gamma}_{\beta b} \tag{11}$$

- Velocity error

$$\delta\vec{v}^{\,\gamma}_{\beta b} = \vec{v}^{\,\gamma}_{\beta b} - \hat{\vec{v}}^{\,\gamma}_{\beta b} \tag{12}$$

- Specific force errors

$$\delta\vec{f}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} - \hat{\vec{f}}^{\,b}_{ib} \tag{13}$$

$$\Delta_e\vec{f}^{\,b}_{ib} = \Delta\vec{f}^{\,b}_{ib} - \Delta\hat{\vec{f}}^{\,b}_{ib} = -\delta\vec{f}^{\,b}_{ib} \tag{14}$$

- Angular rate errors

$$\delta\vec{\omega}^{\,b}_{ib} = \vec{\omega}^{\,b}_{ib} - \hat{\vec{\omega}}^{\,b}_{ib} \tag{15}$$

$$\Delta_e\vec{\omega}^{\,b}_{ib} = \Delta\vec{\omega}^{\,b}_{ib} - \Delta\hat{\vec{\omega}}^{\,b}_{ib} = -\delta\vec{\omega}^{\,b}_{ib} \tag{16}$$

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### Attitude Error Definition

Define

$$\delta C^{\gamma}_{b} = C^{\gamma}_{b}\hat{C}^{b}_{\gamma} = e^{[\delta\vec{\psi}^{\,\gamma}_{\gamma b}\times]} \approx \mathcal{I} + [\delta\vec{\psi}^{\,\gamma}_{\gamma b}\times] \tag{17}$$

This is the error in attitude resulting from errors in estimating the angular rates. .12

### Attitude Error Properties

The attitude error is a multiplicative small angle transformation from the actual frame to the computed frame

$$\hat{C}_b^\gamma = (\mathcal{I} - [\delta\vec{\psi}\,^\gamma_{\gamma b}\times])C_b^\gamma \tag{18}$$

Similarly,

$$C_b^\gamma = (\mathcal{I} + [\delta\vec{\psi}\,^\gamma_{\gamma b}\times])\hat{C}_b^\gamma \tag{19}$$

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### Estimate of Sensor Measurement

Similarly the measuremed specific force and angular rate may be written in terms of the estimates as

$$\tilde{\vec{f}}\,^b_{ib} = \hat{\vec{f}}\,^b_{ib} + \Delta\hat{\vec{f}}\,^b_{ib} \tag{20}$$

$$\tilde{\vec{\omega}}\,^b_{ib} = \hat{\vec{\omega}}\,^b_{ib} + \Delta\hat{\vec{\omega}}\,^b_{ib} \tag{21}$$

where $\hat{\vec{f}}\,^b_{ib}$ and $\hat{\vec{\omega}}\,^b_{ib}$ are the accelerometer and gyroscope estimated calibration values, respectively. .14

# 4 ECI Error Mechanization

### Problem Statement

Since the sensor measurements are corrupted with errors, derive an error model describing the position, velocity, and attitude as a function of time. .15

## 4.1 ECI Frame

### ECI Error Mechanization Attitude

$$\dot{C}_b^i = C_b^i\Omega_{ib}^b \;= \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}\,^i_{ib}\times])\hat{C}_b^i\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}\,^i_{ib}\times])\hat{C}_b^i\Omega_{ib}^b = [\delta\dot{\vec{\psi}}\,^i_{ib}\times]\hat{C}_b^i + (\mathcal{I} + [\delta\vec{\psi}\,^i_{ib}\times])\dot{\hat{C}}_b^i =$$

$$= (\mathcal{I} + [\delta\vec{\psi}\,^i_{ib}\times])\hat{C}_b^i(\hat{\Omega}_{ib}^b + \delta\Omega_{ib}^b) =$$

$$= \hat{C}_b^i\delta\Omega_{ib}^b + (\mathcal{I} + [\delta\vec{\psi}\,^i_{ib}\times])\hat{C}_b^i\hat{\Omega}_{ib}^b$$

$$[\delta\dot{\vec{\psi}}\,^i_{ib}\times] = \hat{C}_b^i\delta\Omega_{ib}^b\hat{C}_i^b = [\hat{C}_b^i\delta\vec{\omega}\,^b_{ib}\times] \tag{22}$$

$$\delta\dot{\vec{\psi}}\,^i_{ib} = \hat{C}_b^i\delta\omega_{ib}^b \tag{23}$$

where $[\delta\vec{\psi}\,^i_{ib}\times]\delta\Omega_{ib}^b \approx 0$. .16

### ECI Error Mechanization Velocity

$$\dot{\vec{v}}\,^i_{ib} = C_b^i\vec{f}\,^b_{ib} + \vec{\gamma}\,^i_{ib} \tag{24}$$

$$\dot{\hat{\vec{v}}}^i_{ib} = \hat{C}_b^i\hat{\vec{f}}\,^b_{ib} + \hat{\vec{\gamma}}^i_{ib} = (\mathcal{I} - [\delta\vec{\psi}\,^i_{ib}\times])C_b^i(\vec{f}\,^b_{ib} + \Delta_e\vec{f}\,^b_{ib}) + \hat{\vec{\gamma}}^i_{ib} \tag{25}$$

$$\begin{aligned}\delta\dot{\vec{v}}\,^i_{ib} = \dot{\vec{v}}\,^i_{ib} - \dot{\hat{\vec{v}}}^i_{ib} &= [\delta\vec{\psi}\,^i_{ib}\times]C_b^i\vec{f}\,^b_{ib} + \hat{C}_b^i\delta\vec{f}\,^b_{ib} + \delta\vec{\gamma}\,^i_{ib}\\ &= [\delta\vec{\psi}\,^i_{ib}\times]\hat{C}_b^i\hat{\vec{f}}\,^b_{ib} + \hat{C}_b^i\delta\vec{f}\,^b_{ib} + \delta\vec{\gamma}\,^i_{ib}\end{aligned}$$

$$\delta\dot{\vec{v}}\,^i_{ib} = -[\hat{C}_b^i\hat{\vec{f}}\,^b_{ib}\times]\delta\vec{\psi}\,^i_{ib} + \hat{C}_b^i\delta\vec{f}\,^b_{ib} + \delta\vec{\gamma}\,^i_{ib} \tag{26}$$

where $\hat{\vec{f}}\,^b_{ib} = \tilde{\vec{f}}\,^b_{ib} - \Delta\hat{\vec{f}}\,^b_{ib} = \vec{f}\,^b_{ib} + \Delta\vec{f}\,^b_{ib} - \Delta\hat{\vec{f}}\,^b_{ib} = \vec{f}\,^b_{ib} + \Delta_e\vec{f}\,^b_{ib} = \vec{f}\,^b_{ib} - \delta\vec{f}\,^b_{ib}$ .17

ECI Error Mechanization  Gravity Error

$$\vec{\gamma}^{\,i}_{ib} \approx \frac{\left(r^e_{eS}(L_b)\right)^2}{(r^e_{eS}(L_b)+h_b)^2} + \vec{\gamma}^{\,i}_0(L_b) \tag{27}$$

Assuming $h_b \ll r^e_{eS}$

$$\delta\vec{\gamma}^{\,i}_{ib} \approx -2\frac{(h_b-\hat{h}_b)}{r^e_{eS}(\hat{L}_b)} g_0(\hat{L}_b)\hat{\vec{u}}^i_D \tag{28}$$

Then converting from curvlinear coordinates to ECI

$$\delta\vec{\gamma}^{\,i}_{ib} \approx \frac{2g_0(\hat{L}_b)}{r^e_{eS}(\hat{L}_b)} \frac{\hat{\vec{r}}^i_{ib}}{|\hat{\vec{r}}^i_{ib}|^2} (\hat{\vec{r}}^i_{ib})^T \delta\vec{r}^{\,i}_{ib} \tag{29}$$

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ECI Error Mechanization  Position

$$\dot{\vec{r}}^{\,i}_{ib} = \vec{v}^{\,i}_{ib} \tag{30}$$

$$\delta\dot{\vec{r}}^{\,i}_{ib} = \delta\vec{v}^{\,i}_{ib} \tag{31}$$

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ECI Error Mechanization  Summary - in terms of $\delta\vec{f}^{\,b}_{ib}, \delta\vec{\omega}^{\,b}_{ib}$

$$\begin{pmatrix} \delta\dot{\vec{\psi}}^{\,i}_{ib} \\ \delta\dot{\vec{v}}^{\,i}_{ib} \\ \delta\dot{\vec{r}}^{\,i}_{ib} \end{pmatrix} = \begin{bmatrix} 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ -[\hat{C}^i_b \hat{\vec{f}}^{\,b}_{ib}\times] & 0_{3\times3} & \frac{2g_0(\hat{L}_b)}{r^e_{eS}(\hat{L}_b)} \frac{\hat{\vec{r}}^i_{ib}}{|\hat{\vec{r}}^i_{ib}|^2} (\hat{\vec{r}}^i_{ib})^T \\ 0_{3\times3} & \mathcal{I}_{3\times3} & 0_{3\times3} \end{bmatrix} \begin{pmatrix} \delta\vec{\psi}^{\,i}_{ib} \\ \delta\vec{v}^{\,i}_{ib} \\ \delta\vec{r}^{\,i}_{ib} \end{pmatrix} + \begin{bmatrix} 0 & \hat{C}^i_b \\ \hat{C}^i_b & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \delta\vec{f}^{\,b}_{ib} \\ \delta\vec{\omega}^{\,b}_{ib} \end{pmatrix} \tag{32}$$

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ECI Error Mechanization  Summary - in terms of $\Delta_e\vec{f}^{\,b}_{ib}, \Delta_e\vec{\omega}^{\,b}_{ib}$

$$\begin{pmatrix} \delta\dot{\vec{\psi}}^{\,i}_{ib} \\ \delta\dot{\vec{v}}^{\,i}_{ib} \\ \delta\dot{\vec{r}}^{\,i}_{ib} \end{pmatrix} = \begin{bmatrix} 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ -[\hat{C}^i_b \hat{\vec{f}}^{\,b}_{ib}\times] & 0_{3\times3} & \frac{2g_0(\hat{L}_b)}{r^e_{eS}(\hat{L}_b)} \frac{\hat{\vec{r}}^i_{ib}}{|\hat{\vec{r}}^i_{ib}|^2} (\hat{\vec{r}}^i_{ib})^T \\ 0_{3\times3} & \mathcal{I}_{3\times3} & 0_{3\times3} \end{bmatrix} \begin{pmatrix} \delta\vec{\psi}^{\,i}_{ib} \\ \delta\vec{v}^{\,i}_{ib} \\ \delta\vec{r}^{\,i}_{ib} \end{pmatrix} + \begin{bmatrix} 0 & -\hat{C}^i_b \\ -\hat{C}^i_b & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \Delta_e\vec{f}^{\,b}_{ib} \\ \Delta_e\vec{\omega}^{\,b}_{ib} \end{pmatrix} \tag{33}$$

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