EE 570: Location and Navigation

On-Line Bayesian Tracking

Aly El-Osery[1] Stephen Bruder[2]

[1]Electrical Engineering Department, New Mexico Tech
Socorro, New Mexico, USA
[2]Electrical and Computer Engineering Department, Embry-Riddle Aeronautical Univesity
Prescott, Arizona, USA

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Sequentially estimate on-line the states of a system as it changes over time using observations that are corrupted with noise.

- *Filtering*: the time of the estimate coincides with the last measurement.
- *Smoothing*: the time of the estimate is within the span of the measurements.
- *Prediction*: the time of the estimate occurs after the last available measurement.

$$\vec{x}_k = \mathbf{f}_k(\vec{x}_{k-1}, \vec{w}_{k-1}) \tag{1}$$

$$\vec{z}_k = \mathbf{h}_k(\vec{x}_k, \vec{v}_k) \tag{2}$$

Given State-Space Equations

$(n \times 1)$ state vector at time k

$$\vec{x}_k = \mathbf{f}_k(\vec{x}_{k-1}, \vec{w}_{k-1}) \tag{1}$$

$$\vec{z}_k = \mathbf{h}_k(\vec{x}_k, \vec{v}_k) \tag{2}$$

$(m \times 1)$ measurement vector at time k

Given State-Space Equations

Possibly non-linear function,
$\mathbf{f}_k : \Re^n \times \Re^{n_w} \mapsto \Re^n$

$$\vec{x}_k = \mathbf{f}_k(\vec{x}_{k-1}, \vec{w}_{k-1}) \quad (1)$$

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i.i.d state noise

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i.i.d measurement noise

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The state process is Markov chain, i.e., $p(\vec{x}_k|\vec{x}_1, \ldots, \vec{x}_{k-1}) = p(\vec{x}_k|\vec{x}_{k-1})$ and the distribution of $\vec{z}_k$ conditional on the state $\vec{x}_k$ is independent of previous state and measurement values, i.e., $p(\vec{z}_k|\vec{x}_{1:k}, \vec{z}_{1:k-1}) = p(\vec{z}_k|\vec{x}_k)$

Objective

Probabilistically estimate $\vec{x}_k$ using previous measurement $\vec{z}_{1:k}$. In other words, construct the pdf $p(\vec{x}_k|\vec{z}_{1:k})$.

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Optimal MMSE Estimate

$$\mathbb{E}\{\|\vec{x}_k - \hat{\vec{x}}_k\|^2|\vec{z}_{1:k}\} = \int \|\vec{x}_k - \hat{\vec{x}}_k\|^2 p(\vec{x}_k|\vec{z}_{1:k})d\vec{x}_k \tag{3}$$

in other words find the conditional mean

$$\hat{\vec{x}}_k = \mathbb{E}\{\vec{x}_k|\vec{z}_{1:k}\} = \int \vec{x}_k p(\vec{x}_k|\vec{z}_{1:k})d\vec{x}_k \tag{4}$$

- $\vec{\boldsymbol{w}}_k$ and $\vec{\boldsymbol{v}}_k$ are drawn from a Gaussian distribution, uncorrelated have zero mean and statistically independent.

$$\mathbb{E}\{\vec{\boldsymbol{w}}_k \vec{\boldsymbol{w}}_i^T\} = \begin{cases} \mathbf{Q}_k & i = k \\ 0 & i \neq k \end{cases} \tag{5}$$

$$\mathbb{E}\{\vec{\boldsymbol{v}}_k \vec{\boldsymbol{v}}_i^T\} = \begin{cases} \mathbf{R}_k & i = k \\ 0 & i \neq k \end{cases} \tag{6}$$

$$\mathbb{E}\{\vec{\boldsymbol{w}}_k \vec{\boldsymbol{v}}_i^T\} = \begin{cases} 0 & \forall i, k \end{cases} \tag{7}$$

- $\mathbf{f}_k$ and $\mathbf{h}_k$ are both linear, e.g., the state-space system equations may be written as

$$\vec{x}_k = \Phi_{k-1}\,\vec{x}_{k-1} + \vec{w}_{k-1} \tag{8}$$

$$\vec{y}_k = \mathbf{H}_k\,\vec{x}_k + \vec{v}_k \tag{9}$$

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($n \times n$) transition matrix relating $\vec{x}_{k-1}$ to $\vec{x}_k$

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$$\vec{\boldsymbol{y}}_k = \mathbf{H}_k \vec{\boldsymbol{x}}_k + \vec{\boldsymbol{v}}_k \tag{9}$$

($\boldsymbol{m} \times \boldsymbol{n}$) matrix provides noiseless connection between measurement and state vectors

State-Space Equations

$$\hat{\vec{x}}_{k|k-1} = \mathbf{\Phi}_{k-1}\hat{\vec{x}}_{k-1|k-1} \quad (10)$$

$$\mathbf{P}_{k|k-1} = \mathbf{Q}_{k-1} + \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{\Phi}_{k-1}^{T} \quad (11)$$

$$\hat{\vec{x}}_{k|k} = \hat{\vec{x}}_{k|k-1} + \mathbf{K}_k \left(\vec{z}_k - \mathbf{H}_k\hat{\vec{x}}_{k|k-1}\right) \quad (12)$$

$$\mathbf{P}_{k|k} = (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\mathbf{P}_{k|k-1} \quad (13)$$

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$(n \times m)$ Kalman gain

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Measurement innovation

Kalman Gain

$$\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}_k^T(\ \mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^T + \mathbf{R}_k\ )^{-1} \tag{14}$$

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Covariance of the innovation term

Initial estimate ($\hat{\vec{x}}_0$ and $\mathbf{P_0}$)

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↓

Compute Kalman gain

$$\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}_k^T(\mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^T + \mathbf{R}_k)^{-1}$$

Kalman filter data flow

Initial estimate ($\hat{\vec{x}}_0$ and $\mathbf{P}_0$)

Compute Kalman gain

$$\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}_k^T(\mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^T + \mathbf{R}_k)^{-1}$$

Update estimate with measurement $\vec{z}_k$

$$\hat{\vec{x}}_{k|k} = \hat{\vec{x}}_{k|k-1} + \mathbf{K}_k(\vec{z}_k - \mathbf{H}_k\hat{\vec{x}}_{k|k-1})$$

Kalman filter data flow

Initial estimate ($\hat{\vec{x}}_0$ and $\mathbf{P}_0$)

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Update error covariance

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{K}_k\mathbf{H}_k\mathbf{P}_{k|k-1}$$

Kalman filter data flow

Initial estimate ($\hat{\vec{x}}_0$ and $\mathbf{P}_0$)

Compute Kalman gain

$$\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}_k^T(\mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^T + \mathbf{R}_k)^{-1}$$

Update estimate with measurement $\vec{z}_k$

$$\hat{\vec{x}}_{k|k} = \hat{\vec{x}}_{k|k-1} + \mathbf{K}_k(\vec{z}_k - \mathbf{H}_k\hat{\vec{x}}_{k|k-1})$$

Update error covariance

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{K}_k\mathbf{H}_k\mathbf{P}_{k|k-1}$$

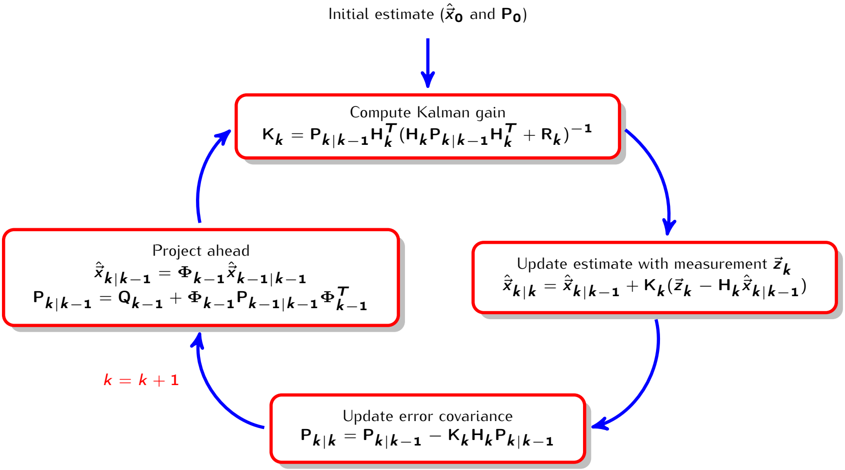
$k = k + 1$

Project ahead

$$\hat{\vec{x}}_{k|k-1} = \mathbf{\Phi}_{k-1}\hat{\vec{x}}_{k-1|k-1}$$

$$\mathbf{P}_{k|k-1} = \mathbf{Q}_{k-1} + \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{\Phi}_{k-1}^T$$

Kalman filter data flow



If R is a block matrix, i.e., $R = diag(R^1, R^2, \ldots, R^r)$. The R^i has dimensions $p^i \times p^i$. Then, we can sequentially process the measurements as:
For $i = 1, 2, \ldots, r$

$$\mathbf{K}^i = \mathbf{P}^{i-1}(\mathbf{H}^i)^T(\mathbf{H}^i\mathbf{P}^{i-1}(\mathbf{H}^i)^T + \mathbf{R}^i)^{-1} \tag{15}$$

$$\hat{\vec{x}}^i_{k|k} = \hat{\vec{x}}^i_{k|k} + \mathbf{K}^i(\vec{z}^i_k - \mathbf{H}^i\hat{\vec{x}}^{i-1}_{k|k}) \tag{16}$$

$$\mathbf{P}^i = (\mathbf{I} - \mathbf{K}^i\mathbf{H}^i)\mathbf{P}^{i-1} \tag{17}$$

where $\hat{\vec{x}}^0_{k|k} = \hat{\vec{x}}_{k|k-1}$, $\mathbf{P}^0 = \mathbf{P}^0_{k|k-1}$ and $\mathbf{H}^i$ is $p^i \times n$ corresponding to the rows of $\mathbf{H}$ corresponding the measurement being processed.

The system is observable if the observability matrix

$$\mathcal{O}(k) = \begin{bmatrix} \boldsymbol{H}(k-n+1) \\ \boldsymbol{H}(k-n-2)\boldsymbol{\Phi}(k-n+1) \\ \vdots \\ \boldsymbol{H}(k)\boldsymbol{\Phi}(k-1)\dots\boldsymbol{\Phi}(k-n+1) \end{bmatrix} \tag{18}$$

where n is the number of states, has a rank of n. The rank of $\mathcal{O}$ is a binary indicator and does **not** provide a measure of how close the system is to being unobservable, hence, is prone to numerical ill-conditioning.

In addition to the computation of the rank of $\mathcal{O}(k)$, compute the Singular Value Decomposition (SVD) of $\mathcal{O}(k)$ as

$$\mathcal{O} = U \Sigma V^* \tag{19}$$

and observe the diagonal values of the matrix Σ. Using this approach it is possible to monitor the variations in the system observability due to changes in system dynamics.

- Kalman filter is optimal under the aforementioned assumptions,
- and it is also an unbiased and minimum variance estimate.
- If the Gaussian assumptions is not true, Kalman filter is biased and not minimum variance.
- Observability is dynamics dependent.
- The error covariance update may be implemented using the *Joseph form* which provides a more stable solution due to the guaranteed symmetry.

$$\mathbf{P}_{k|k} = (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\,\mathbf{P}_{k|k-1}\,(\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)^T + \mathbf{K}_k\mathbf{R}_k\mathbf{K}_k^T \tag{20}$$

$$\dot{\vec{x}}(t) = \mathbf{F}(t)\vec{x}(t) + \mathbf{G}(t)\vec{w}(t) \tag{21}$$

To obtain the state vector estimate $\hat{\vec{x}}(t)$

$$\mathbb{E}\{\dot{\vec{x}}(t)\} = \frac{\partial}{\partial t}\hat{\vec{x}}(t) = \mathbf{F}(t)\hat{\vec{x}}(t) \tag{22}$$

Solving the above equation over the interval $t - \tau_s, t$

$$\hat{\vec{x}}(t) = e^{\left(\int_{t-\tau_s}^{t} \mathbf{F}(t')dt'\right)}\hat{\vec{x}}(t - \tau_s) \tag{23}$$

where $\mathbf{F}_{k-1}$ is the average of $\mathbf{F}$ at times t and $t - \tau_s$.

As shown in the Kalman filter equations the state vector estimate is given by

$$\hat{\vec{\boldsymbol{x}}}_{k|k-1} = \boldsymbol{\Phi}_{k-1}\hat{\vec{\boldsymbol{x}}}_{k-1|k-1}$$

Therefore,

As shown in the Kalman filter equations the state vector estimate is given by

$$\hat{\vec{x}}_{k|k-1} = \mathbf{\Phi}_{k-1}\hat{\vec{x}}_{k-1|k-1}$$

Therefore,

$$\mathbf{\Phi}_{k-1} = e^{\mathbf{F}_{k-1}\tau_s} \approx \mathbf{I} + \mathbf{F}_{k-1}\tau_s \tag{24}$$

where $\mathbf{F}_{k-1}$ is the average of $\mathbf{F}$ at times t and $t - \tau_s$, and first order approximation is used.

Assuming white noise, small time step, **G** is constant over the integration period, and the trapezoidal integration

$$\mathbf{Q}_{k-1} \approx \frac{1}{2}\left[\mathbf{\Phi}_{k-1}\mathbf{G}_{k-1}\mathbf{Q}(t_{k-1})\mathbf{G}_{k-1}^T\mathbf{\Phi}_{k-1}^T + \mathbf{G}_{k-1}\mathbf{Q}(t_{k-1})\mathbf{G}_{k-1}^T\right]\tau_s \quad (25)$$

where

$$\mathbb{E}\{\vec{\boldsymbol{w}}(\eta)\vec{\boldsymbol{w}}^T(\zeta)\} = \mathbf{Q}(\eta)\delta(\eta-\zeta) \quad (26)$$

$$\mathbf{F}_k = \left.\frac{\partial \mathbf{f}(\vec{x})}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}_{k|k-1}}, \qquad \mathbf{H}_k = \left.\frac{\partial \mathbf{h}(\vec{x})}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}_{k|k-1}} \tag{27}$$

where

$$\frac{\partial \mathbf{f}(\vec{x})}{\partial \vec{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots \\ \vdots & \ddots & \vdots \end{pmatrix}, \qquad \frac{\partial \mathbf{h}(\vec{x})}{\partial \vec{x}} = \begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots \\ \vdots & \ddots & \vdots \end{pmatrix} \tag{28}$$

State Equation

$$\dot{b}(t) = -\frac{1}{T_c} b(t) + w(t) \tag{29}$$

Autocorrelation Function

$$\mathbb{E}\{b(t)b(t+\tau)\} = \sigma_{BI}^2 e^{-|\tau|/T_c} \tag{30}$$

where

$$\mathbb{E}\{w(t)w(t+\tau)\} = Q(t)\delta(t-\tau) \tag{31}$$

$$Q(t) = \frac{2\sigma_{BI}^2}{T_c} \tag{32}$$

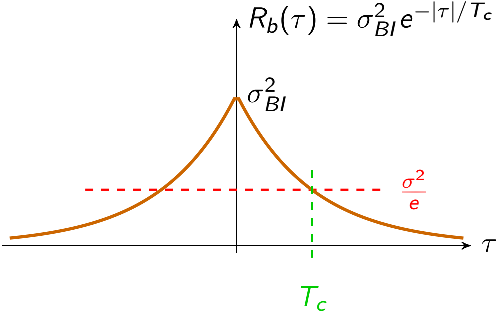
and T_c is the correlation time.

State Equation

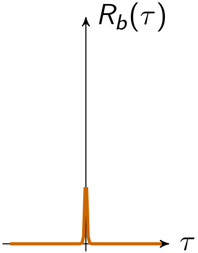
$$b_k = e^{-\frac{1}{T_c}\tau_s} b_{k-1} + w_{k-1} \tag{33}$$

System Covariance Matrix

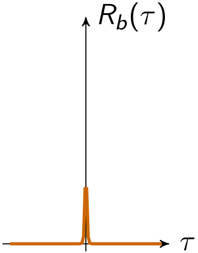
$$Q = \sigma_{BI}^2 [1 - e^{-\frac{2}{T_c}\tau_s}] \tag{34}$$

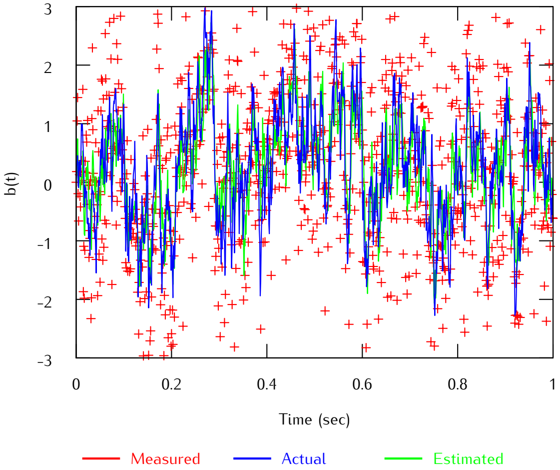

$R_b(\tau) = \sigma_{BI}^2 e^{-|\tau|/T_c}$
σ_{BI}^2
$\frac{\sigma^2}{e}$
τ
T_c

Small Correlation Time $T_c = 0.01$

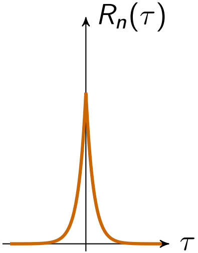


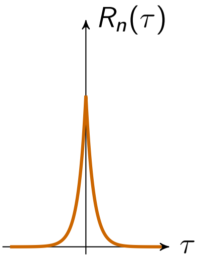
Small Correlation Time $T_c = 0.01$



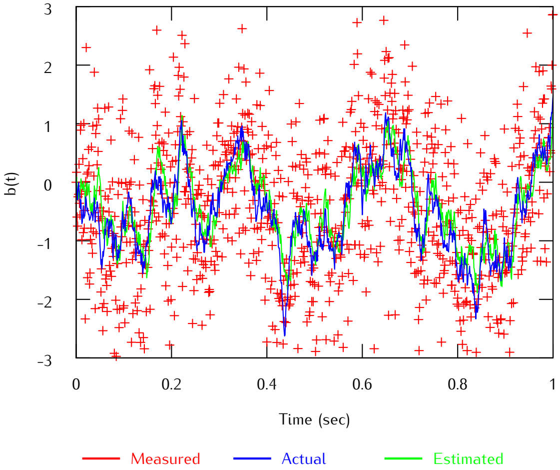


Larger Correlation Time $T_c = 0.1$





$R_n(\tau)$
τ



3
2
1
0
-1
-2
-3
b(t)
0
0.2
0.4
0.6
0.8
1
Time (sec)
Measured
Actual
Estimated

Propagates carefully chosen sample points (using unscented transformation) through the true non-linear system, and therefore captures the posterior mean and covariance accurately to the second order.

A Monte Carlo based method. It allows for a complete representation of the state distribution function. Unlike EKF and UKF, particle filters do not require the Gaussian assumptions.

Bayesian Filtering: From Kalman Filters to Particle Filters, and Beyond,
by Zhe Chen