# EE 570: Location and Navigation

## Navigation Mathematics: Kinematics (Coordinate Frame Transformation)

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- Determine the detailed kinematic relationships between the 4 major frames of interest
  - The Earth-Centered Inertial (ECI) coordinate frame (*i*-frame)
  - The Earth-Centered Earth-Fixed (ECEF) coordinate frame (*e*-frame)
  - The Local Navigation (Nav) coordinate frame (*n*-frame)
  - The Body coordinate frame (*b*-frame)

- Relationship between the ECI and ECEF frames
  - ECI & ECEF have co-located orgins

$$\vec{r}_{ie} = \dot{\vec{r}}_{ie} = \ddot{\vec{r}}_{ie} = 0$$

  - The $x$, $y$, and $z$ axis of the ECI & ECEF frames are coincident at time $t_0$
  - The ECEF frame rotates about the common $z$-axis at a fixed rate ($\omega_{ie}$)
    - Ignoring minor speed variatins (precession & nutation) $\omega_{ie} = 72.921151467\mu$rad/sec (WGS84) which is $\approx 15^\circ$/hr

- The angular velocity and acceleration are

$$\vec{\omega}^{\,i}_{ie} = \begin{bmatrix} 0 \\ 0 \\ \omega_{ie} \end{bmatrix} \qquad \dot{\vec{\omega}}^{\,i}_{ie} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- The angle of rotation is

$$\begin{aligned} \theta_{ie} &= \omega_{ie}(t - t_0) \\ &= \omega_{ie} t + \theta_{GMST} \end{aligned}$$

where GMST is the Greenwich mean sidereal time
- The orientation of frame $\{e\}$ *wrt* frame $\{i\}$ becomes

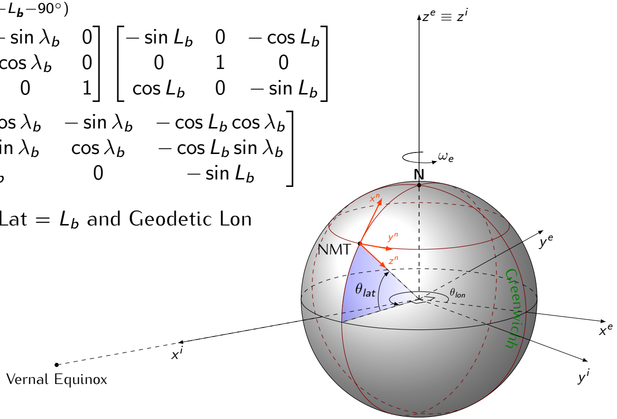
$$C^i_e = R_{(\vec{z},\theta_{ie})} = \begin{bmatrix} \cos\theta_{ie} & -\sin\theta_{ie} & 0 \\ \sin\theta_{ie} & \cos\theta_{ie} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: $\vec{\omega}^{\,i}_{ie} = \vec{\omega}^{\,e}_{ie}$

- Description of the navigation frame
  - Orientation of the *n*-frame *wrt* the *e*-frame

$$
\begin{aligned}
C_n^e &= R_{(\vec{z},\lambda_b)} R_{(\vec{y},-L_b-90^\circ)} \\
&= \begin{bmatrix} \cos\lambda_b & -\sin\lambda_b & 0 \\ \sin\lambda_b & \cos\lambda_b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin L_b & 0 & -\cos L_b \\ 0 & 1 & 0 \\ \cos L_b & 0 & -\sin L_b \end{bmatrix} \\
&= \begin{bmatrix} -\sin L_b \cos\lambda_b & -\sin\lambda_b & -\cos L_b \cos\lambda_b \\ -\sin L_b \sin\lambda_b & \cos\lambda_b & -\cos L_b \sin\lambda_b \\ \cos L_b & 0 & -\sin L_b \end{bmatrix}
\end{aligned}
$$

where geodetic Lat $= L_b$ and Geodetic Lon $= \lambda_b$



- Angular velocity of the *n*-frame *wrt* the *e*-frame resolved in the *e*-frame as a skew-symmetric matrix

$$\dot{C}_n^e = C_n^e \Omega_{en}^n = \Omega_{en}^e C_n^e$$

$$\begin{aligned}
\Omega_{en}^e &= \dot{C}_n^e \, [C_n^e]^T \\
&= \begin{bmatrix} s_{L_b} s_{\lambda_b} \dot{\lambda}_b - c_{L_b} c_{\lambda_b} \dot{L}_b & -c_{\lambda_b} \dot{\lambda}_b & -c_{\lambda_b} s_{L_b} \dot{L}_b + c_{L_b} s_{\lambda_b} \dot{\lambda}_b \\ -c_{L_b} s_{\lambda_b} \dot{L}_b - c_{\lambda_b} s_{L_b} \dot{\lambda}_b & -s_{\lambda_b} \dot{\lambda}_b & s_{L_b} s_{\lambda_b} \dot{L}_b - c_{L_b} c_{\lambda_b} \dot{\lambda}_b \\ -s_{L_b} \dot{L}_b & 0 & -c_{L_b} \dot{L}_b \end{bmatrix} [C_n^e]^T \\
&= \begin{bmatrix} 0 & -\dot{\lambda}_b & -\dot{L}_b \cos(\lambda_b) \\ \dot{\lambda}_b & 0 & -\dot{L}_b \sin(\lambda_b) \\ \dot{L}_b \cos(\lambda_b) & \dot{L}_b \sin(\lambda_b) & 0 \end{bmatrix}
\end{aligned}$$

- The angular velocity vector

$$\vec{\omega}\,^{e}_{en} = \begin{bmatrix} \sin(\lambda)\dot{L}_b \\ -\cos(\lambda)\dot{L}_b \\ \dot{\lambda}_b \end{bmatrix} \qquad \vec{\omega}\,^{n}_{en} = [C^e_n]^T\, \omega^e_{en} = \begin{bmatrix} \cos(\lambda)\dot{\lambda}_b \\ -\dot{L}_b \\ -\sin(L_b)\dot{\lambda}_b \end{bmatrix}$$

- Hence the orientation of the *n*-frame *wrt* the *i*-frame becomes

$$C_n^i = C_e^i C_n^e = \begin{bmatrix} c_{\theta_{ie}} & -s_{\theta_{ie}} & 0 \\ s_{\theta_{ie}} & c_{\theta_{ie}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s_{L_b} c_{\lambda_b} & -s_{\lambda_b} & -c_{L_b} c_{\lambda_b} \\ -s_{L_b} s_{\lambda_b} & c_{\lambda_b} & -c_{L_b} s_{\lambda_b} \\ -c_{L_b} & 0 & -s_{L_b} \end{bmatrix}$$

$$= \begin{bmatrix} -\sin(L_b)\cos(\theta_{ie}+\lambda_b) & -\sin(\theta_{ie}+\lambda_b) & -\cos(L_b)\cos(\theta_{ie}+\lambda_b) \\ -\sin(L_b)\sin(\theta_{ie}+\lambda_b) & \cos(\theta_{ie}+\lambda_b) & -\cos(L_b)\sin(\theta_{ie}+\lambda_b) \\ \cos(L_b) & 0 & -\sin(L_b) \end{bmatrix}$$

- The angular velocity of the *n*–frame *wrt* the *i*–frame resolved in the *i*–frame is

$$
\begin{aligned}
\vec{\omega}^{\,i}_{in} &= \vec{\omega}^{\,i}_{ie} + C^i_e \vec{\omega}^{\,e}_{en} \\
&= \begin{bmatrix} 0 \\ 0 \\ \omega_{ie} \end{bmatrix} + \begin{bmatrix} \cos(\theta_{ie}) & -\sin(\theta_{ie}) & 0 \\ \sin(\theta_{ie}) & \cos(\theta_{ie}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin(\lambda_b)\dot{L}_b \\ -\cos(\lambda_b)\dot{L}_b \\ \dot{\lambda}_b \end{bmatrix} \\
&= \begin{bmatrix} \sin(\theta_{ie} + \lambda_b)\dot{L}_b \\ -\cos(\theta_{ie} + \lambda_b)\dot{L}_b \\ \omega_{ie} + \dot{\lambda}_b \end{bmatrix}
\end{aligned}
$$

- The vector from the origin of the $e$–frame to the $n$–frame origin resolved in the $e$–frame (from the last lecture)

$$\vec{r}^{\,e}_{eb} = \begin{bmatrix} (R_E + h_b)\cos(L_b)\cos(\lambda_b) \\ (R_E + h_b)\cos(L_b)\sin(\lambda_b) \\ (R_E(1-e^2) + h_b)\sin(L_b) \end{bmatrix} = \vec{r}^{\,e}_{en}$$

Origins of the $n$–frame and the $b$–frame are the same

- The velocity of the $n$–frame *wrt* the $e$–frame resolved in the $e$–frame

$$\vec{v}^{\,e}_{en} = \frac{d}{dt}\vec{r}^{\,e}_{en} = \frac{\partial \vec{r}^{\,e}_{en}}{\partial L_b}\dot{L}_b + \frac{\partial \vec{r}^{\,e}_{en}}{\partial \lambda_b}\dot{\lambda}_b + \frac{\partial \vec{r}^{\,e}_{en}}{\partial h_b}\dot{h}_b$$

$$= \begin{bmatrix} -\sin(L_b)\cos(\lambda_b) & -\sin(\lambda_b) & -\cos(L_b)\cos(\lambda_b) \\ -\sin(L_b)\sin(\lambda_b) & \cos(\lambda_b) & -\cos(L_b)\sin(\lambda_b) \\ \cos(L_b) & 0 & -\sin(L_b) \end{bmatrix} \begin{bmatrix} (R_N + h_b)\dot{L}_b \\ \cos(L_b)(R_E + h_b)\dot{\lambda}_b \\ -\dot{h}_b \end{bmatrix}$$

- Recalling the form of $C_n^e$ suggests that

$$\vec{v}^{\,e}_{en} = C_n^e \begin{bmatrix} (R_N + h_b)\dot{L}_b \\ \cos(L_b)(R_E + h_b)\dot{\lambda}_b \\ -\dot{h}_b \end{bmatrix} = C_n^e \vec{v}^{\,n}_{en}$$

- and hence,

$$\vec{v}^{\,n}_{en} = \begin{bmatrix} (R_N + h_b)\dot{L}_b \\ \cos(L_b)(R_E + h_b)\dot{\lambda}_b \\ -\dot{h}_b \end{bmatrix}$$

- Restating $\vec{v}^{\,n}_{en}$ as

$$\vec{v}^{\,n}_{en} = \begin{bmatrix} (R_N + h_b)\dot{L}_b \\ \cos(L_b)(R_E + h_b)\dot{\lambda}_b \\ -\dot{h}_b \end{bmatrix} = \begin{bmatrix} \vec{v}^{\,n}_{en,N} \\ \vec{v}^{\,n}_{en,E} \\ \vec{v}^{\,n}_{en,D} \end{bmatrix}$$

- and recalling that

$$\vec{\omega}^{\,n}_{en} = \begin{bmatrix} \cos(L_b)\dot{\lambda}_b \\ -\dot{L}_b \\ -\sin(L_b)\dot{\lambda}_b \end{bmatrix}$$

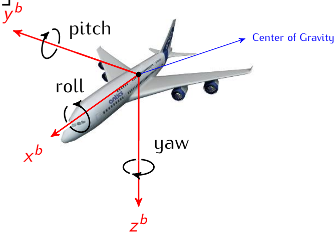
- Suggests that

$$\vec{\omega}^{\,n}_{en} = \begin{bmatrix} \vec{v}^{\,n}_{en,E}/(R_E + h_b) \\ -\vec{v}^{\,n}_{en,N}/(R_N + h_b) \\ -\tan(L_b)\vec{v}^{\,n}_{en,E}/(R_E + h_b) \end{bmatrix}$$

- Description *wrt* the body frame
  - Orientation of the *b*-frame *wrt* the *n*-frame in terms of relative yaw ($\psi$), pitch ($\theta$), then roll ($\phi$) angles

$$C_b^n = R_{(\vec{z},\psi)} R_{(\vec{y},\theta)} R_{(\vec{x},\phi)} = \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & -s_\phi \\ 0 & s_\phi & c_\phi \end{bmatrix}$$

$$= \begin{bmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & c_\phi c_\psi s_\theta + s_\phi s_\psi \\ c_\theta s_\psi & c_\phi c_\psi + s_\theta s_\phi s_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}$$



- The angular velocity of the *b*-frame *wrt* the *i*-frame resolved/coordinatized in the *i*-frame

$$\begin{aligned}\vec{\omega}\,^{i}_{ib} &= \vec{\omega}\,^{i}_{in} + C^{i}_{n}\vec{\omega}\,^{n}_{nb} \\ &= \vec{\omega}\,^{i}_{ie} + C^{i}_{e}\vec{\omega}\,^{e}_{en} + C^{i}_{n}\vec{\omega}\,^{n}_{nb}\end{aligned}$$

- Position vectors to the origin of the body frame
  - The origins of the body and Nav frames are co-incident

$$\vec{r}_{nb} = \vec{0}$$

  - The origins of the ECI and ECEF frames are co-incident

$$\vec{r}_{eb} = \vec{r}_{ib} = \vec{r}_{en} = \vec{r}_{in}$$

  - Velocity of the *b*-frame *wrt* the *i*-frame resolved in the *i*-frame
    - Case #2: A moving point in a rotation frame

$$\begin{aligned} \vec{v}^{\,i}_{ib} &= \frac{d}{dt}\vec{r}^{\,i}_{ib} = \frac{d}{dt}C^i_e\vec{r}^{\,e}_{eb} \\ &= C^i_e\Omega^e_{ie}\vec{r}^{\,e}_{eb} + C^i_e\vec{v}^{\,e}_{eb} \\ &= C^i_e\left(\Omega^e_{ie}\vec{r}^{\,e}_{eb} + \vec{v}^{\,e}_{eb}\right) \end{aligned}$$

- Acceleration of the *b*-frame *wrt* the *i*-frame resolved in the *i*-frame
  - Case #2: A moving point in a rotation frame

$$\vec{a}^{\,i}_{ib} = \frac{d}{dt}\vec{v}^{\,i}_{ib} = \frac{d}{dt}\left(C^i_e\left(\Omega^e_{ie}\vec{r}^{\,e}_{eb} + \vec{v}^{\,e}_{eb}\right)\right)$$

$$= \dot{C}^i_e\left(\Omega^e_{ie}\vec{r}^{\,e}_{eb} + \vec{v}^{\,e}_{eb}\right) + C^i_e\left(\overset{\dot{\omega}=0}{\cancel{\dot{\Omega}^e_{ie}}}\,\vec{r}^{\,e}_{eb} + \Omega^e_{ie}\dot{\vec{r}}^{\,e}_{eb} + \dot{\vec{v}}^{e}_{eb}\right)$$

$$= C^i_e\Omega^e_{ie}\left(\Omega^e_{ie}\vec{r}^{\,e}_{eb} + \vec{v}^{\,e}_{eb}\right) + C^i_e\left(\Omega^e_{ie}\vec{v}^{\,e}_{eb} + \vec{a}^{\,e}_{eb}\right)$$

$$= C^i_e\left(\Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb} + 2\Omega^e_{ie}\vec{v}^{\,e}_{eb} + \vec{a}^{\,e}_{eb}\right)$$