Lecture

Navigation Mathematics: Kinematics (Earth Surface & Gravity Models)

EE 570: Location and Navigation

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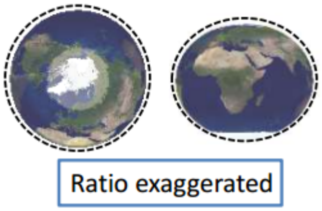
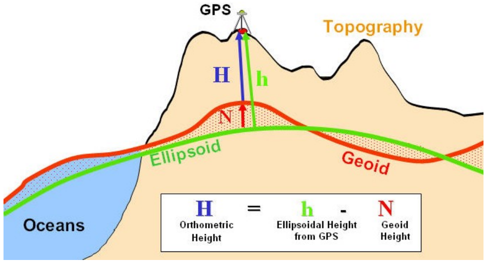
Earth Modeling

- The earth can be modeled as an oblate spheroid
  - A circular cross section when viewed from the polar axis (top view)
  - An elliptical cross-section when viewed perpendicular to the polar axis (side view)



- This ellipsoid (i.e., oblate spheroid) is an approximation of the "geoid"
- The geoid is a gravitational equipotential surface which "best" fits (in the least square sense) the mean sea level



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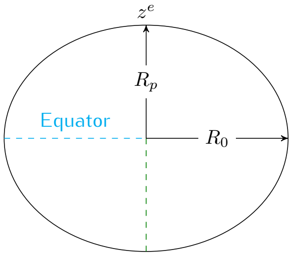
Earth Modeling

- WGS 84 provides as model of the earth's geoid
  - More recently replace by EGM 2008
- The equatorial radius radius $R_0 = 6,378,137.0\text{m}$
- The polar radius radius $R_p = 6,356,752.3142\text{m}$
- Eccentricity of the ellipsoid

$$e = \sqrt{1 - \frac{R_p^2}{R_0^2}} \approx 0.0818$$

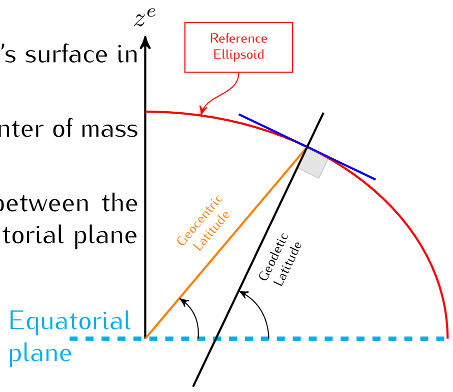
- Flattening of the ellipsoid

$$f = \frac{R_0 - R_p}{R_0} \approx \frac{1}{298}$$

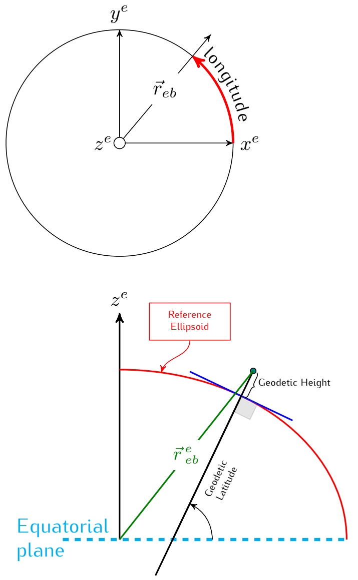


Earth Modeling

- We can define a position “near” the earth’s surface in terms of latitude, longitude, and height
    - Geocentric latitude intersects the center of mass of the earth
    - Geodetic latitude (L) is the angle between the normal to the ellipsoid and the equatorial plane



Earth Modeling



- The longitude (λ) is the angle from the x-axis of the ECEF frame to the projection of $\vec{r}_{eb}$ onto the equatorial plane
- The geodetic (or ellipsoidal) height (h) is the distance along the normal from the ellipsoid to the body

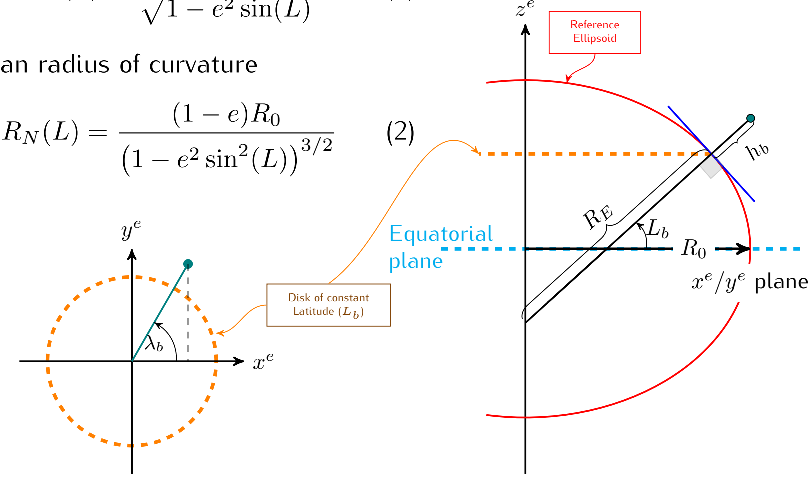
Earth Modeling

- Transverse radius of curvature

$$R_E(L) = \frac{R_0}{\sqrt{1 - e^2 \sin(L)}} \tag{1}$$

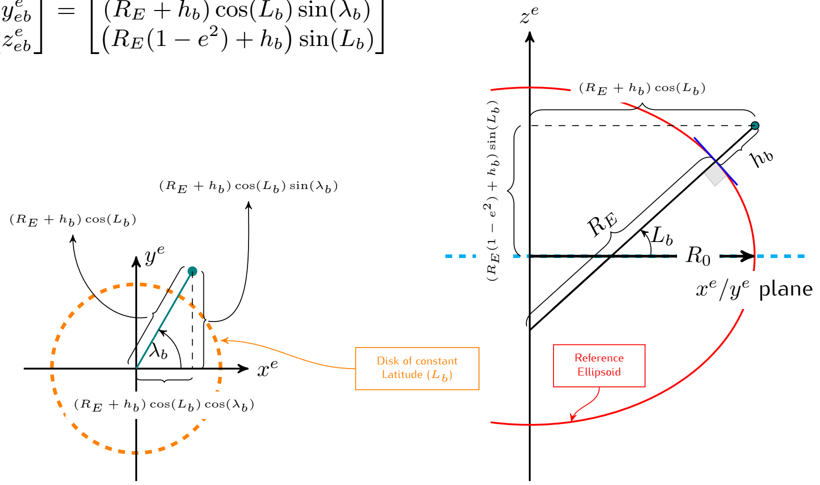
- Meridian radius of curvature

$$R_N(L) = \frac{(1-e)R_0}{\left(1 - e^2 \sin^2(L)\right)^{3/2}} \tag{2}$$



Earth Modeling

$$\vec r^{\,e}_{eb} = \begin{bmatrix} x^e_{eb} \\ y^e_{eb} \\ z^e_{eb} \end{bmatrix} = \begin{bmatrix} (R_E + h_b)\cos(L_b)\cos(\lambda_b) \\ (R_E + h_b)\cos(L_b)\sin(\lambda_b) \\ \left(R_E(1-e^2) + h_b\right)\sin(L_b) \end{bmatrix}$$



Gravity Models

- Specific force ($\vec f_{ib}$)
  - Non-gravitational force per unit mass (unit of acceleration)
    - Accelerometers measure specific force
- Specific force sensed when stationary (*wrt* earth) is referred to as the acceleration due to gravity ($\vec g_b$)
  - Actually, the reaction to this force
- Gravitational force (γ_{ib}) is result of mass attraction
  - The gravitational mass attraction force is different from the acceleration due to gravity

Gravity Models

- Relationship between specific force, inertial acceleration, and gravitational attraction

$$\vec f_{ib} \equiv \vec a_{ib} - \vec \gamma_{ib} \tag{3}$$

- When stationary on the surface of the earth
  - Recall case 1: A fixed point in a rotating frame

$$\ddot{\vec r}^{\,0}_{02}(t) = \overset{\dot\omega = 0}{\cancel{\dot{\vec\omega}^{\,0}_{01} \times \vec r^{\,0}_{12}(t)}} \quad + \vec\omega^{\,0}_{01} \times \left(\vec\omega^{\,0}_{01} \times \vec r^{\,0}_{12}(t)\right)$$

  - Consider frame {0} to be the {i} frame, {1}={e}, and {2}={b} gives

$$\ddot{\vec r}^{\,i}_{ib}(t) = \vec\omega^{\,i}_{ie} \times \left(\vec\omega^{\,i}_{ie} \times \vec r^{\,i}_{eb}(t)\right)$$

  - coordinatizing in the e-frame

$$\ddot{\vec r}^{\,e}_{ib}(t) = \vec\omega^{\,e}_{ie} \times (\vec\omega^{\,e}_{ie} \times \vec r^{\,e}_{eb}(t))$$

Gravity Models

- Thus, when stationary on the surface of the earth the acceleration is due to centrifugal force

$$\vec{a}^{\,e}_{ib} = \Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb}$$

- Therefore, the acceleration due to gravity is

$$\vec{g}^{\,e}_{b} = -\left.\vec{f}_{ib}\right|_{\vec{v}^{\,e}_{eb}=0} = -\Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb} + \vec{\gamma}^{\,e}_{ib} \tag{4}$$

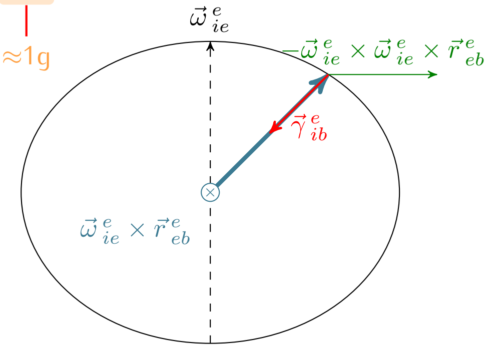
Specific force

$$\vec{f}_{ib} = \vec{a}_{ib} - \vec{\gamma}_{ib} \tag{5}$$

Gravity Models

- Vector diagram

$$\underbrace{\vec{g}^{\,e}_{b}}_{1\text{g}} = \underbrace{-\Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb}}_{\approx 3\text{mg}} + \underbrace{\vec{\gamma}^{\,e}_{ib}}_{\approx 1\text{g}}$$



Gravity Models

- Now, $\vec{\omega}^{\,e}_{ie} = \begin{bmatrix}0\\0\\1\end{bmatrix}\omega_{ie}$ and hence, $\Omega^e_{ie} = \begin{bmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 0\end{bmatrix}\omega_{ie}$, and thus

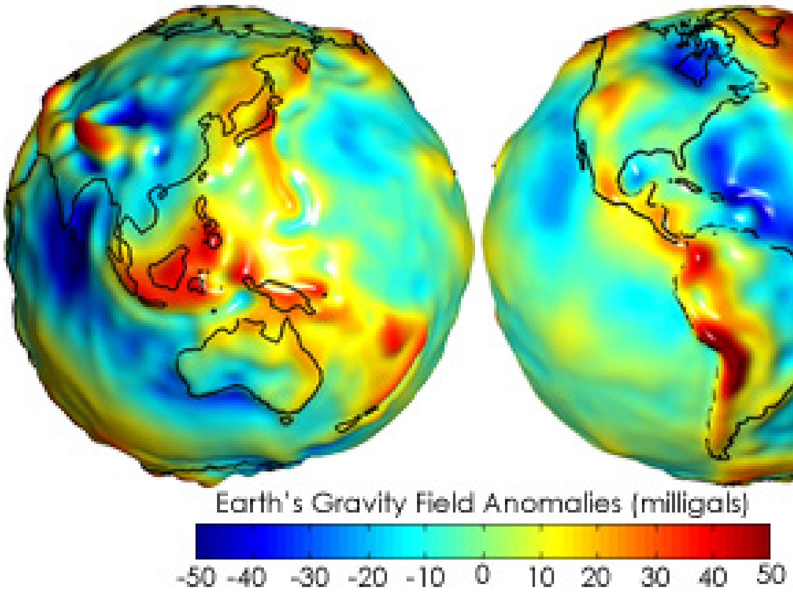
$$\vec{g}^{\,e}_{b} = \vec{\gamma}^{\,e}_{ib} + \omega^2_{ie}\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 0\end{bmatrix}\vec{r}^{\,e}_{eb}$$

- The WGS 84 model of acceleration due to gravity (on the ellipsoid) can be approximated by (Somigliana model)

$$g_0(L) = 9.7803253359\frac{\left(1 + 0.001931853\sin^2(L)\right)}{\sqrt{1 - e^2\sin^2(L)}} \tag{6}$$

Gravity Models

- On March 17, 2002 NASA launched the Gravity Recovery and Climate Experiment (GRACE) which led to the development of some of the most precise Earth gravity models.


Earth's Gravity Field Anomalies (milligals)
-50 -40 -30 -20 -10 0 10 20 30 40 50

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