EE 570: Location and Navigation

Kalman Filtering Example

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- Kalman filter (KF) is optimal under the assumptions that the system is linear and the noise is uncorrelated
- Under these assumptions KF provides an unbiased and minimum variance estimate.
- If the Gaussian assumptions is not true, Kalman filter is biased and not minimum variance.
- If the noise is correlated we can augment the states of the system to maintain the the uncorrelated requirement of the system noise.

Given a state space system

$$\dot{\vec{x}}_1(t) = F_1(t)\vec{x}_1(t) + G_1(t)\vec{w}_1(t)$$

$$\vec{y}_1(t) = H_1(t)\vec{x}_1(t) + \vec{v}(t)$$

As we have seen the noise $\vec{w}_1(t)$ non-white, e.g., correlated Gaussian noise, and as such may be modeled as

$$\dot{\vec{x}}_2(t) = F_2(t)\vec{x}_2(t) + G_2(t)\vec{w}_2(t)$$

$$\vec{w}_1(t) = H_2(t)\vec{x}_2(t)$$

Define a new augmented state

$$\vec{x}_{aug} = \begin{pmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{pmatrix} \quad (1)$$

therefore,

$$\dot{\vec{x}}_{aug} = \begin{pmatrix} \dot{\vec{x}}_1(t) \\ \dot{\vec{x}}_2(t) \end{pmatrix} = \begin{pmatrix} F_1(t) & G_1 H_2(t) \\ 0 & F_2 \end{pmatrix} \begin{pmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ G_2(t) \end{pmatrix} \vec{w}_2(t) \quad (2)$$

and

$$\vec{y}(t) = \begin{pmatrix} H_1 & 0 \end{pmatrix} \begin{pmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{pmatrix} + \vec{v}(t) \quad (3)$$

Correlated Measurement Noise

Given a state space system

$$\dot{\vec{x}}_1(t) = F_1(t)\vec{x}_1(t) + G_1(t)\vec{w}(t)$$

$$\vec{y}_1(t) = H_1(t)\vec{x}_1(t) + \vec{v}_1(t)$$

In this case the measurement noise $\vec{v}_1$ may be correlated

$$\dot{\vec{x}}_2(t) = F_2(t)\vec{x}_2(t) + G_2(t)\vec{v}_2(t)$$

$$\vec{v}_1(t) = H_2(t)\vec{x}_2(t)$$

Define a new augmented state

$$\vec{x}_{aug} = \begin{pmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{pmatrix} \tag{4}$$

therefore,

$$\dot{\vec{x}}_{aug} = \begin{pmatrix} \dot{\vec{x}}_1(t) \\ \dot{\vec{x}}_2(t) \end{pmatrix} = \begin{pmatrix} F_1(t) & 0 \\ 0 & F_2 \end{pmatrix} \begin{pmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{pmatrix} + \begin{pmatrix} G_1(t) & 0 \\ 0 & G_2(t) \end{pmatrix} \begin{pmatrix} \vec{w}(t) \\ \vec{v}_2 \end{pmatrix} \tag{5}$$

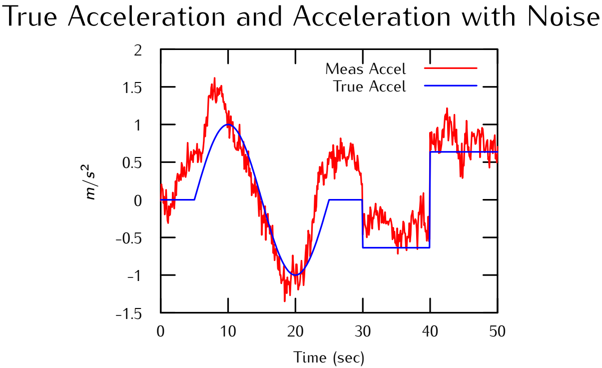
and

$$\vec{y}(t) = \begin{pmatrix} H_1 & H_2 \end{pmatrix} \begin{pmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{pmatrix} \tag{6}$$

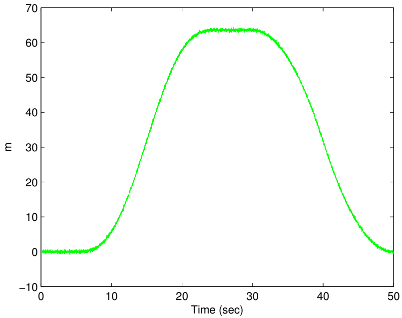
You are to design a system that estimates the position and velocity of a moving point in a straight line. You have:

1. an accelerometer corrupted with noise
2. an aiding sensor allowing you to measure absolute position that is also corrupted with noise.

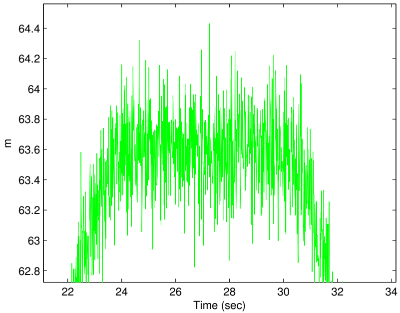
- Sampling Rate Fs = 100Hz.
- Accelerometer specs
  1. VRW $= 1mg/\sqrt{Hz}$.
  2. BI $= 7mg$ with correlation time 6s.
- Position measurement is corrupted with WGN. $\sim \mathcal{N}(0, \sigma_p^2)$, where $\sigma_p = 2.5$m


True Acceleration and Acceleration with Noise
Meas Accel
True Accel
m/s^2
Time (sec)
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
10
20
30
40
50

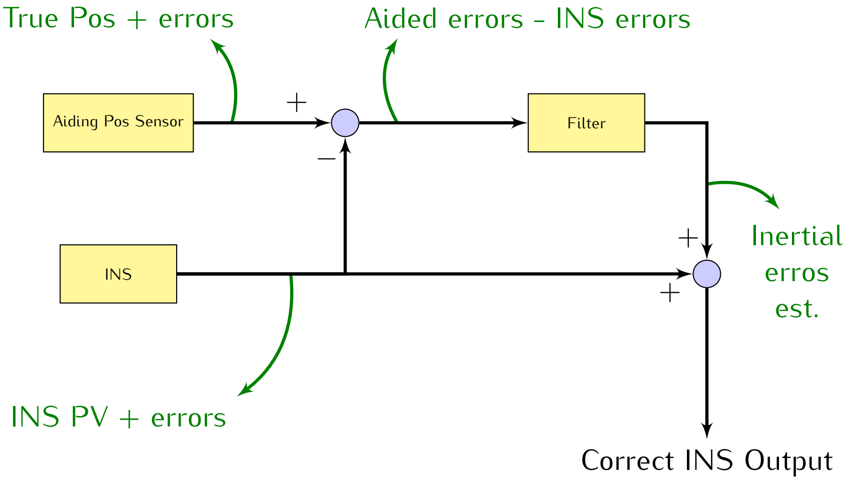
Absolute position measurment corrupted with noise



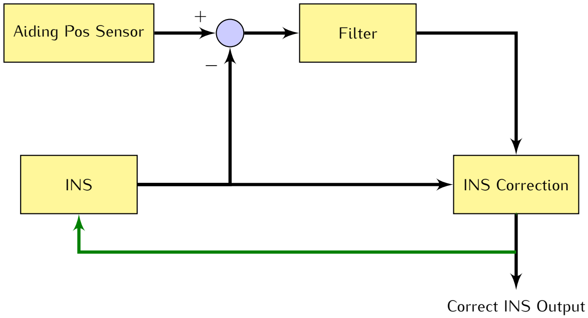
zoomed version



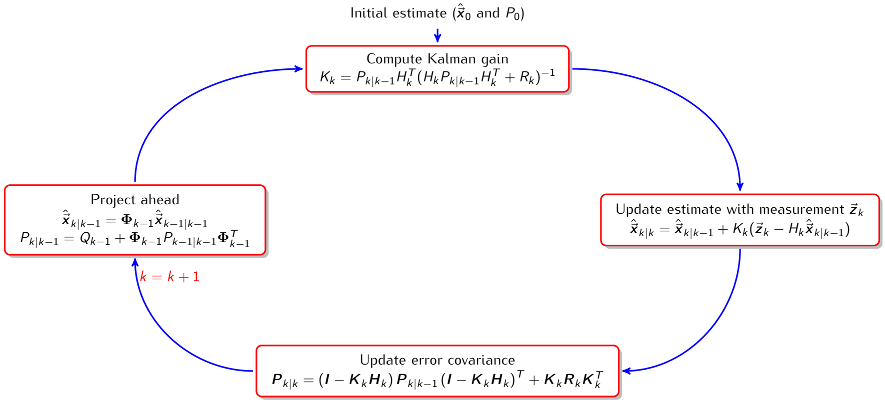
1. Clean up the noisy input to the system.
2. Use Kalamn filtering techiques with
   - A model of the system dyanmics (too ristrictive)
   - A model of the error dyanmics and correct the system output in
     - open-loop configuration, or
     - closed-loop configuration.


True Pos + errors
Aided errors - INS errors
Aiding Pos Sensor
+
−
Filter
Inertial
erros
est.
INS
+
+
INS PV + errors
Correct INS Output

If error estimates are fedback to correct the INS mechanization, a reset of the state estimates becomes necessary.



Kalman filter data flow



- State noise covariance matrix (continuous)

$$\mathbb{E}\{\vec{w}(t)\hat{\vec{w}}^T(\tau)\} = Q(t)\delta(t-\tau)$$

- State noise covariance matrix (discrete)

$$\mathbb{E}\{\vec{w}_k\hat{\vec{w}}_i^T\} = \begin{cases} Q_k & i = k \\ 0 & i \neq k \end{cases}$$

- Measurement noise covariance matrix

$$\mathbb{E}\{\vec{v}_k\hat{\vec{v}}_i^T\} = \begin{cases} R_k & i = k \\ 0 & i \neq k \end{cases}$$

- Error covariance matrix

$$P_k = \mathbb{E}\{(\vec{x}_k - \hat{\vec{x}}_k)(\vec{x}_k - \hat{\vec{x}}_k)^T\} = \mathbb{E}\{\vec{e}_k\hat{\vec{e}}_k^T\}$$

The position, velocity and acceleration may be modeled using the following kinematic model.

$$\begin{aligned} \dot{p}(t) &= v(t) \\ \dot{v}(t) &= a(t) \end{aligned} \tag{7}$$

where $a(t)$ is the input.

Assuming that the accelerometer sensor measurement may be modeled as

$$\tilde{a}(t) = a(t) + b(t) + w_a(t) \tag{8}$$

and the bias is Markov, therefore

$$\dot{b}(t) = -\frac{1}{T_c} b(t) + w_b(t) \tag{9}$$

where both $w_a(t)$ and $w_b(t)$ are WGN. In addition we have the position measurment from which we can derive a delta position

$$\delta p(t) = p(t) - \hat{p}(t) \tag{10}$$

where $\hat{p}$ is the derived position by double integrating the measured acceleration.

Define the position and velocity error terms as

$$\begin{aligned} \delta\dot{p}(t) &= \dot{p}(t) - \dot{\hat{p}}(t) \\ &= v(t) - \hat{v}(t) \\ &= \delta v(t) \end{aligned} \tag{11}$$

and

$$\begin{aligned} \delta\dot{v}(t) &= \dot{v}(t) - \dot{\hat{v}}(t) \\ &= a(t) - \hat{a}(t) \\ &= -b(t) - w_a(t) \end{aligned} \tag{12}$$

where $b(t)$ is modeled as shown in Eq. 9

First Order Markov Noise

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State Equation

$$\dot{b}(t) = -\frac{1}{T_c} b(t) + w_b(t) \tag{13}$$

Autocorrelation Function

$$\mathbb{E}\{b(t)b(t+\tau)\} = \sigma^2 e^{-|\tau|/T_c} \tag{14}$$

where

$$\mathbb{E}\{w_b(t)w_b(t+\tau)\} = Q_b(t)\delta(t-\tau) \tag{15}$$

$$Q_b(t) = \frac{2\sigma_{BI}^2}{T_c} \tag{16}$$

and T_c is the correlation time and σ_{BI} is the bias instability.

State Space Formulation

$$\dot{\vec{x}}(t) = \begin{pmatrix} \delta\dot{p}(t) \\ \delta\dot{v}(t) \\ \dot{b}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -\frac{1}{T_c} \end{pmatrix} \begin{pmatrix} \delta p(t) \\ \delta v(t) \\ b(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ w_a(t) \\ w_b(t) \end{pmatrix}$$
$$= F(t)\vec{x}(t) + G(t)\vec{w}(t) \tag{17}$$

- The continuous state noise covariance matrix $Q(t)$ is

$$Q(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & VRW^2 & 0 \\ 0 & 0 & \frac{2\sigma_{BI}^2}{Tc} \end{pmatrix} \quad (18)$$

  make sure that the VRW and σ_{BI} are converted to have SI units.
- The measurement noise covariance matrix is $R = \sigma_p^2$

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Now we are ready to start the implementation but first we have to discretize the system.

$$\vec{x}(k+1) = \Phi(k)\vec{x}(k) + \vec{w}_d \tag{19}$$

where

$$\Phi(k) \approx \mathcal{I} + Fdt \tag{20}$$

with the measurement equation

$$y(k) = H\vec{x} + w_p(k) = \delta p(k) + w_p(k) \tag{21}$$

where $H = [1\ 0\ 0]$. The discrete Q_d is approximated as

$$Q_{k-1} \approx \frac{1}{2}[\Phi_{k-1}G(t_{k-1})Q(t_{k-1})G^T(t_{k-1}))\Phi_{k-1}^T + G(t_{k-1})Q(t_{k-1})G^T(t_{k-1})]dt \tag{22}$$

Discrete First Order Markov Noise

State Equation

$$b_k = e^{-\frac{1}{T_c}dt} b_{k-1} + w_{bd,k-1} \tag{23}$$

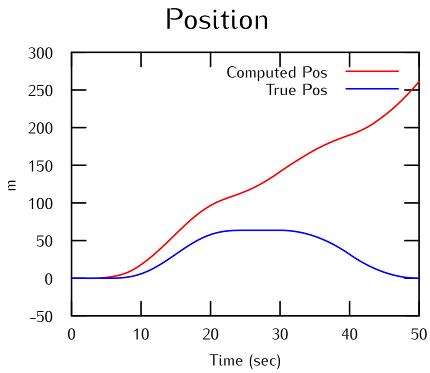
System Covariance Matrix

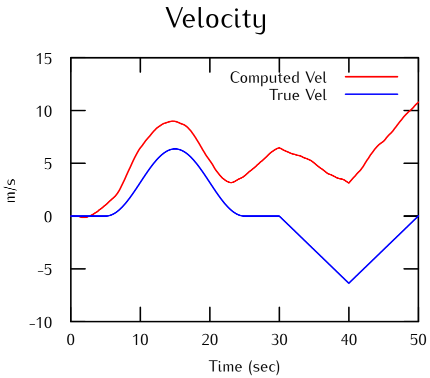
$$Q_{bd} = \sigma_{BI}^2 [1 - e^{-\frac{2}{T_c}dt}] \tag{24}$$

where w_{bd} is the discrete noise for the bias.

You should use Eqs. 23 and 24 to overwrite their values in Φ and Q_d since they don't need to be approximated.

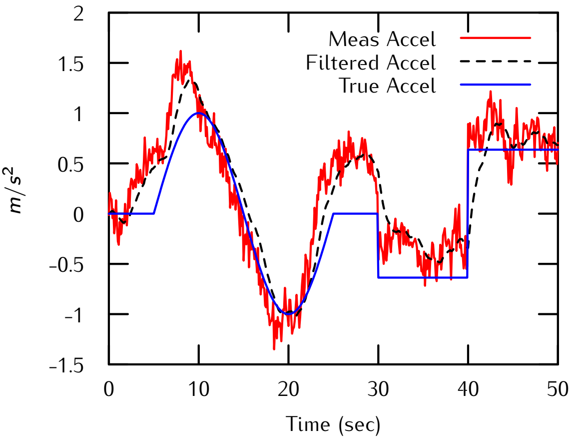
Using only the acceleration measurement and an integration approach to compute the velocity, then integrate again to get position.



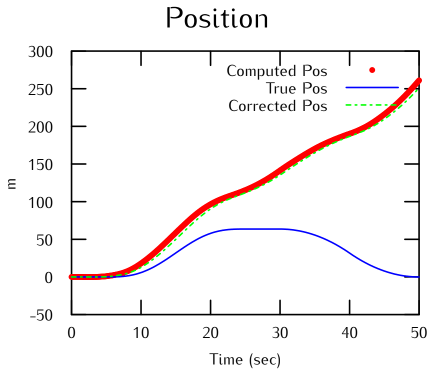


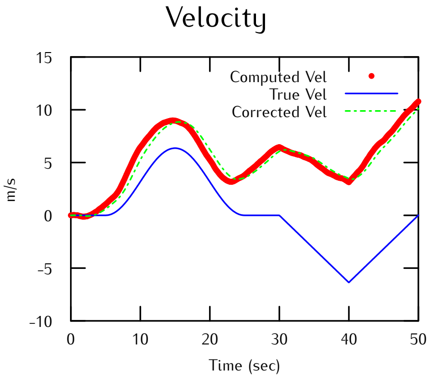
Approach 1 — Filtered input

Filtered Accel Measurement

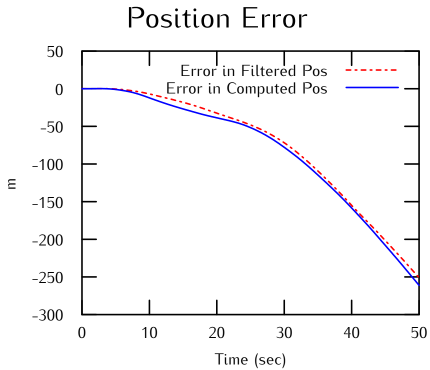


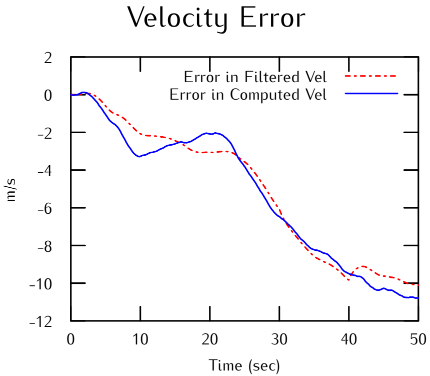
Position and Velocity





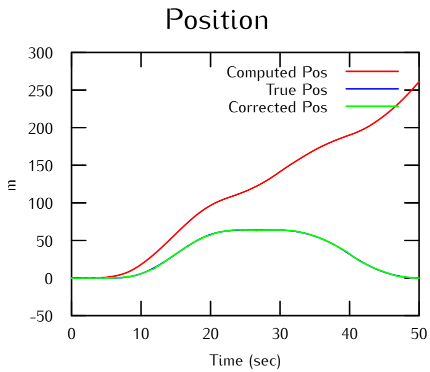
Approach 1 — Filtered input
Position and Velocity Errors

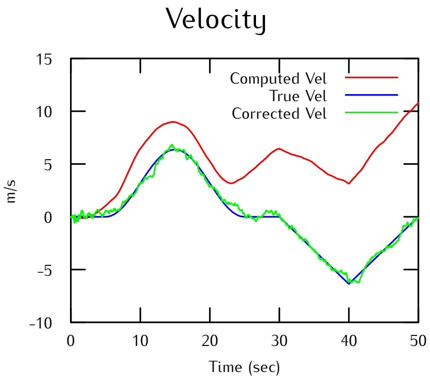


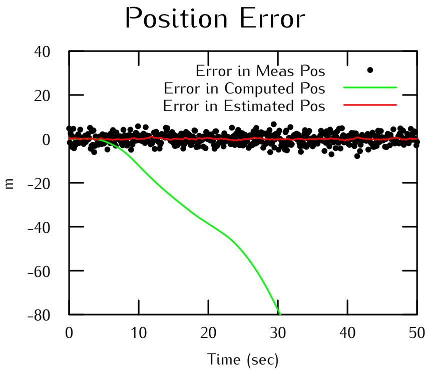


Open-loop Correction

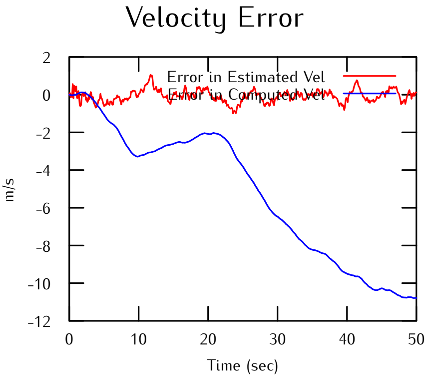
Best estimate = INS out (pos & vel) + KF est error (pos & vel)







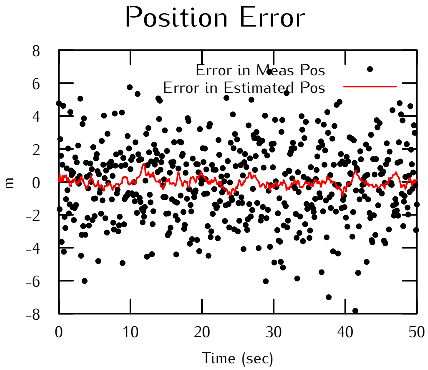
Position Error
Error in Meas Pos
Error in Computed Pos
Error in Estimated Pos
m
40
20
0
-20
-40
-60
-80
0
10
20
30
40
50
Time (sec)

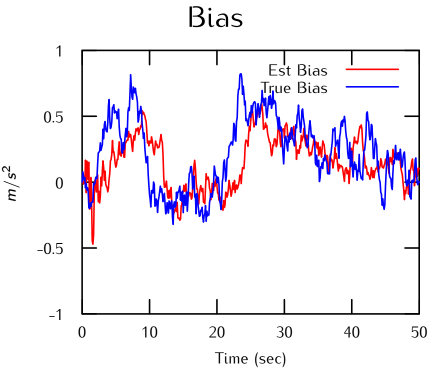


Velocity Error
Error in Estimated Vel
Error in Computed Vel
m/s
2
0
-2
-4
-6
-8
-10
-12
0
10
20
30
40
50
Time (sec)

Approach 2 — Open-Loop Compensation

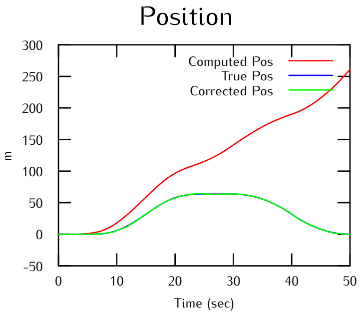
Pos Error & Bias Estimate

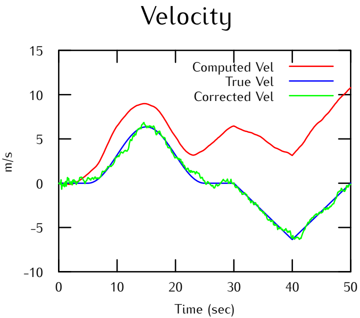


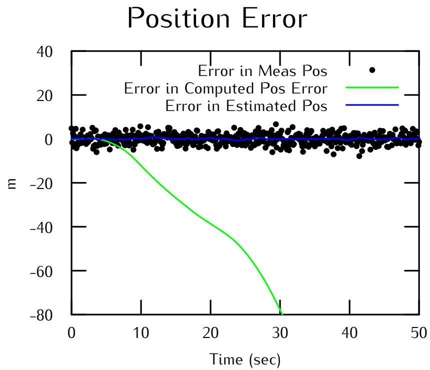


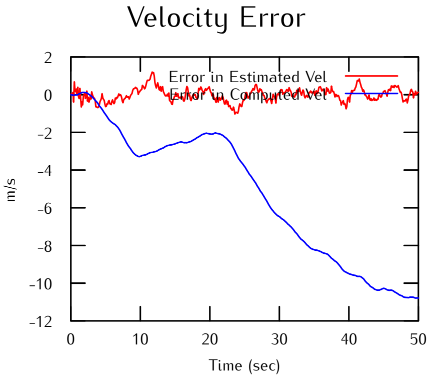
Closed-loop Correction

Best estimate = INS out (pos,vel, & bias) + KF est error (pos, vel & bias)
Use best estimate on next iteration of INS
Accel estimate = accel meas - est bias
Reset state estimates before next call to KF






Position Error
Error in Meas Pos
Error in Computed Pos Error
Error in Estimated Pos
m
40
20
0
-20
-40
-60
-80
0
10
20
30
40
50
Time (sec)


Velocity Error
Error in Estimated Vel
Error in Computed Vel
m/s
2
0
-2
-4
-6
-8
-10
-12
0
10
20
30
40
50
Time (sec)

Pos Error & Bias Estimate