EE 570: Location and Navigation

Navigation Mathematics: Kinematics (Rotation Matrices)

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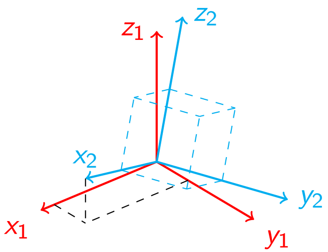
- The notation $\vec{x}_2^1$ (a vector) describes the vector $\vec{x}_2$ in terms (i.e. with respect to) of the "1" frame
  $\Rightarrow$ Resolved or coordinatized in the "1" frame
- The notation C_2^1 (a rotation matrix) describes the "2" coordinate frame (i.e., the $\vec{x}_2$, $\vec{y}_2$, and $\vec{z}_2$ basis vectors) in terms of (i.e., with respect to) the "1" frame
- **Right Hand coordinate frames will be used in this course**

Rotation Matrices

- The rotation matrix C_2^1 describes the orientation of frame $\{2\}$ relative to frame $\{1\}$
- $C_2^1 = \left[\vec{x}\,_2^1, \vec{y}\,_2^1, \vec{z}\,_2^1\right]$



- The vectors $\vec{x}_2$, $\vec{y}_2$, and $\vec{z}_2$ described in coordinate system $\{1\}$ is

$$\vec{x}_2^{\,1} = \begin{bmatrix} <\vec{x}_2, \vec{x}_1> \\ <\vec{x}_2, \vec{y}_1> \\ <\vec{x}_2, \vec{z}_1> \end{bmatrix}, \quad \vec{y}_2^{\,1} = \begin{bmatrix} <\vec{y}_2, \vec{x}_1> \\ <\vec{y}_2, \vec{y}_1> \\ <\vec{y}_2, \vec{z}_1> \end{bmatrix}, \quad \vec{z}_2^{\,1} = \begin{bmatrix} <\vec{z}_2, \vec{x}_1> \\ <\vec{z}_2, \vec{y}_1> \\ <\vec{z}_2, \vec{z}_1> \end{bmatrix},$$

  respectively
- Therefore,

$$C_2^1 = \begin{bmatrix} <\vec{x}_2, \vec{x}_1>, & <\vec{y}_2, \vec{x}_1>, & <\vec{z}_2, \vec{x}_1> \\ <\vec{x}_2, \vec{y}_1>, & <\vec{y}_2, \vec{y}_1>, & <\vec{z}_2, \vec{y}_1> \\ <\vec{x}_2, \vec{z}_1>, & <\vec{y}_2, \vec{z}_1>, & <\vec{z}_2, \vec{z}_1> \end{bmatrix} \quad (1)$$

- By describing frame "1" in terms of frame "2" in order to construct C_1^2, it can be shown that
  1. i.e., sitting on frame {2} looking frame {1}

$$C_1^2 = \left[C_2^1\right]^T = \left[C_2^1\right]^{-1} \tag{2}$$

  2. Note that these rotational matrices are orthonormal, i.e., $C^{-1} = C^T$ and $|C| = \pm 1$
     - For right-hand coordinate systems $|C| = +1$

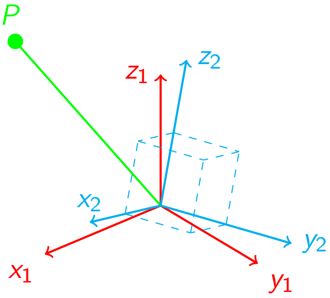
Rotation Matrices as transformation

- The point P can be described in the coordinate system frame $\{1\}$ as

$$\vec{p}^{\,1} = p_{x_1}\vec{x}_1 + p_{y_1}\vec{y}_1 + p_{z_1}\vec{z}_1$$

- The point P can also be described in the coordinate system frame $\{2\}$ as

$$\vec{p}^{\,2} = p_{x_2}\vec{x}_2 + p_{y_2}\vec{y}_2 + p_{z_2}\vec{z}_2$$



Rotation Matrices as transformation (cont.)

- Since $\vec{p}^{\,1}$ and $\vec{p}^{\,2}$ are representation of the same point but *wrt* different coordinate systems, then

$$
\begin{aligned}
p_{x_1} &=< \vec{p}^{\,1}, \vec{x}_1 > \\
&=< \vec{p}^{\,2}, \vec{x}_1 > \\
&=< p_{x_2}\vec{x}_2 + p_{y_2}\vec{y}_2 + p_{z_2}\vec{z}_2, \vec{x}_1 > \\
&= p_{x_2} < \vec{x}_2, \vec{x}_1 > + p_{y_2} < \vec{y}_2, \vec{x}_1 > + p_{z_2} < \vec{z}_2, \vec{x}_1 >
\end{aligned}
$$

- Similarly,

$$
\begin{aligned}
p_{y_1} &= p_{x_2} < \vec{x}_2, \vec{y}_1 > + p_{y_2} < \vec{y}_2, \vec{y}_1 > + p_{z_2} < \vec{z}_2, \vec{y}_1 > \\
p_{z_1} &= p_{x_2} < \vec{x}_2, \vec{z}_1 > + p_{y_2} < \vec{y}_2, \vec{z}_1 > + p_{z_2} < \vec{z}_2, \vec{z}_1 >
\end{aligned}
$$

- Putting it all together

$$\begin{bmatrix} p_{x_1} \\ p_{y_1} \\ p_{z_1} \end{bmatrix} = \begin{bmatrix} <\vec{x}_2, \vec{x}_1>, & <\vec{y}_2, \vec{x}_1>, & <\vec{z}_2, \vec{x}_1> \\ <\vec{x}_2, \vec{y}_1>, & <\vec{y}_2, \vec{y}_1>, & <\vec{z}_2, \vec{y}_1> \\ <\vec{x}_2, \vec{z}_1>, & <\vec{y}_2, \vec{z}_1>, & <\vec{z}_2, \vec{z}_1> \end{bmatrix} \begin{bmatrix} p_{x_2} \\ p_{y_2} \\ p_{z_2} \end{bmatrix} \Rightarrow \vec{p}^{\,1} = C_2^1 \vec{p}^{\,2}$$

which matches Eq. 1

- It can be shown that

$$\vec{p}^{\,2} = C_1^2 \vec{p}^{\,1} = \left[C_2^1\right]^T \vec{p}^{\,1}$$

- Consequently,

$$\left[C_2^1\right]^T = \left[C_2^1\right]^{-1} = C_1^2$$

Rotation matrices can be thought of in three distinct ways:

1. It describes the orientation of one coordinate frame *wrt* another coordinate frame
2. It represents a coordinate transformation relating the coordinates of a point (e.g., P) in two different frames of reference
3. It is an operator taking a vector $\vec{p}$ and rotating it into a new vector $R\vec{p}$, both in the same system