EE 570: Location and Navigation

Navigation Mathematics: Kinematics (Rotation Matrices: Fixed vs Relative)

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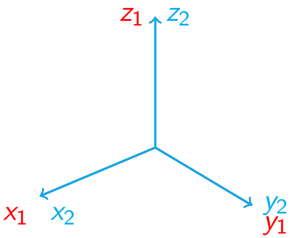
- The rotation matrix C is a 3×3 matrix requiring 9 parameters to describe and orientation.
- It can be shown that any orientation can by uniquely described using only 3-parameters.
- Some examples of 3-parameter descriptions include:
  - Fixed axis rotations
  - Relative (or Euler) axis rotations, and
  - angle-axis rotation

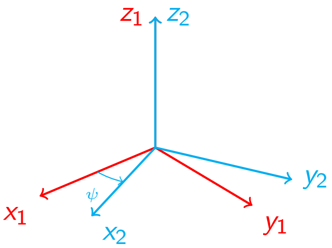
- An example
  - Rotate frame {2} by ψ about the common z-axis.



- An example
  - Rotate frame {2} by ψ about the common z-axis.

$$\vec{x}_2 = \cos(\psi)\vec{x}_1 + \sin(\psi)\vec{y}_1 + 0\vec{z}_1$$

$$\vec{y}_2 = -sin(\psi)\vec{x}_1 + \cos(\psi)\vec{y}_1 + 0\vec{z}_1$$



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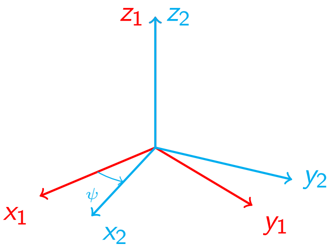
$$C_2^1 = \begin{bmatrix} \cos(\psi) & -\sin(\psi) & 0 \\ \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



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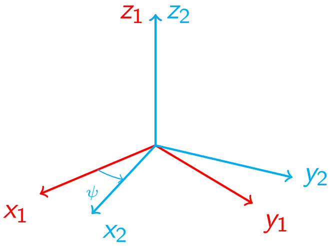


Note

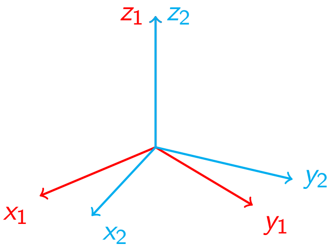
Could have rotated the frame {1} by $-\psi$ to get $C_1^2 = R_{(\vec{z},-\psi)}$

Alternatively,

$$< \vec{x}_2, \vec{x}_1 > = \vec{x}_2 \cdot \vec{x}_1 = |\vec{x}_2||\vec{x}_1| \cos(\psi) = \cos(\psi)$$
$$< \vec{x}_2, \vec{y}_1 > = \cos(90^\circ - \psi) = \sin(\psi)$$
$$< \vec{x}_2, \vec{z}_1 > = \cos(90) = 0$$

$$< \vec{y}_2, \vec{x}_1 > = \cos(90^\circ - \psi)$$
$$< \vec{y}_2, \vec{y}_1 > = \cos(\psi)$$
$$< \vec{y}_2, \vec{z}_1 > = \cos(90) = 0$$

$$< \vec{z}_2, \vec{x}_1 > = \cos(90)$$
$$< \vec{z}_2, \vec{y}_1 > = \cos(90)$$
$$< \vec{z}_2, \vec{z}_1 > = \cos(0) = 1$$



- Recall that

$$C_2^1 = \begin{bmatrix} < \vec{x}_2, \vec{x}_1 >, & < \vec{y}_2, \vec{x}_1 >, & < \vec{z}_2, \vec{x}_1 > \\ < \vec{x}_2, \vec{y}_1 >, & < \vec{y}_2, \vec{y}_1 >, & < \vec{z}_2, \vec{y}_1 > \\ < \vec{x}_2, \vec{z}_1 >, & < \vec{y}_2, \vec{z}_1 >, & < \vec{z}_2, \vec{z}_1 > \end{bmatrix}$$

$$= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{(\vec{z},\psi)}$$

- Similarly,

$$R_{(\vec{x},\phi)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \quad R_{(\vec{y},\theta)} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Consider

$$R_{(\vec{y},\theta)}R_{(\vec{z},\psi)} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta\cos\psi & -\cos\theta\sin\psi & \sin\theta \\ \sin\psi & \cos\psi & 0 \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{bmatrix}$$

Consider

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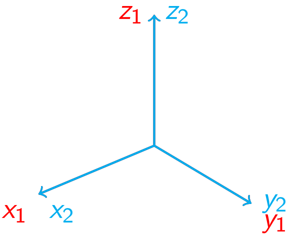
$$\begin{bmatrix} \cos\theta\cos\psi & -\cos\theta\sin\psi & \sin\theta \\ \sin\psi & \cos\psi & 0 \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{bmatrix}$$

$$R_{(\vec{z},\psi)}R_{(\vec{y},\theta)} = \begin{bmatrix} \cos\psi\cos\theta & -\sin\psi & \cos\psi\sin\theta \\ \sin\psi\cos\theta & \cos\psi & \sin\psi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \neq R_{(\vec{y},\theta)}R_{(\vec{z},\psi)}$$

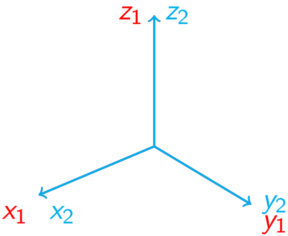
- Fixed axis rotation
  - conduct rotations about the original (i.e., fixed) x–, y–, or z-axis
- Relative axis rotation
  - Conduct rotations about the current location (i.e., relative) of x–, y–, or z-axis
    - Sometimes referred to as Euler rotations

- **Step 1:** Rotate about the z-axis by ψ, then
- **Step 2:** Rotate about the y-axis by θ, then
- **Step 3:** Rotate about the x-axis by ϕ.

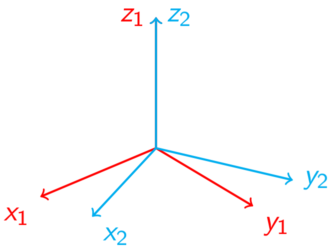
Relative Axis Rotation



Fixed Axis Rotation



Relative Axis Rotation
Rotate about z_1



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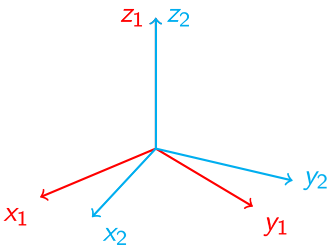
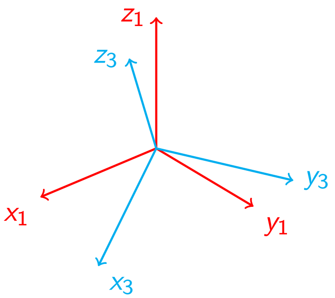


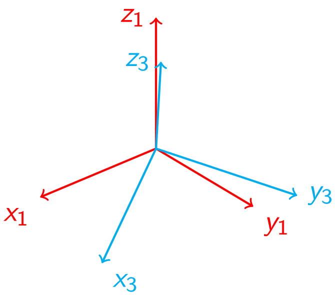
Relative Axis Rotation

Rotate about y_2

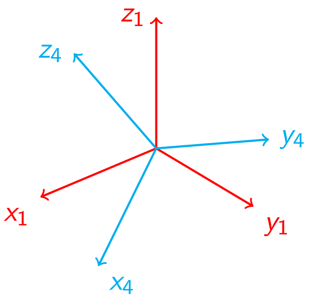


Fixed Axis Rotation

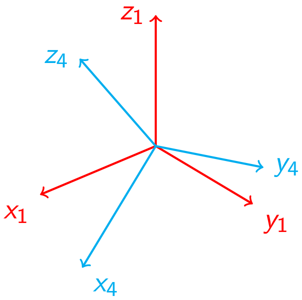
Rotate about y_1



Relative Axis Rotation
Rotate about x_3



Fixed Axis Rotation
Rotate about x_1



Relative Axis Rotations

- It is intuitive to think in terms of the orientation of one coordinate *wrt* another coordinate frame
  - In frame {3} frame {4} is described as $C_4^3 = [\vec{x}_4^{\,3}, \vec{y}_4^{\,3}, \vec{z}_4^{\,3}] = R_{(\vec{x},\phi)}$

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  - In frame {2} frame {3} is described as $C_3^2 = [\vec{x}\,_3^2, \vec{y}\,_3^2, \vec{z}\,_3^2] = R_{(\vec{y},\theta)}$
    - Therefore, in frame {2}, frame {4} is $C_4^2 = C_3^2 C_4^3$

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    - Therefore, in frame {2}, frame {4} is $C_4^2 = C_3^2 C_4^3$
  - In frame {1} frame {2} is described as $C_2^1 = [\vec{x}\,_2^1, \vec{y}\,_2^1, \vec{z}\,_2^1] = R_{(\vec{z},\psi)}$
    - Therefore, in frame {1}, frame {4} is $C_4^1 = C_2^1 C_3^2 C_4^3$

- In the case of relative axis: Yaw (ψ), Pitch (θ), Roll (ϕ) rotation sequence

$$
\begin{aligned}
C_4^1 &= C_2^1 C_3^2 C_4^3 \\
&= R_{(\vec{z},\psi)} R_{(\vec{y},\theta)} R_{(\vec{x},\phi)} \\
&= \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \\
&= \begin{bmatrix} c_\theta c_\psi & c_\psi s_\theta s_\phi - c_\phi s_\psi & c_\phi c_\psi s_\theta + s_\phi s_\psi \\ c_\theta s_\psi & c_\phi c_\psi + s_\theta s_\phi s_\psi & c_\phi s_\theta s_\psi - c_\psi s_\phi \\ -s_\theta & c_\theta s_\phi & c_\theta c_\phi \end{bmatrix}
\end{aligned}
\tag{1}
$$

- It is intuitive to think in terms of taking a vector $\vec{p}$ and rotating it into a new vector $R\vec{p}$ both in the same coordinate frame
  - The first "yaw" rotation, "rotates" the frame {1} basis vector to become the frame {2} basis vector
    $[\vec{x}\,^1_2, \vec{y}\,^1_2, \vec{z}\,^1_2] = R_{(\vec{z},\psi)}[\vec{x}\,^1_1, \vec{y}\,^1_1, \vec{z}\,^1_1] = C^1_2$

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    - The second "pitch" rotation, "rotates" the frame {2} basis vector to become the frame {3} basis vector
      $[\vec{x}\,^1_3, \vec{y}\,^1_3, \vec{z}\,^1_3] = R_{(\vec{y},\theta)}[\vec{x}\,^1_2, \vec{y}\,^1_2, \vec{z}\,^1_2] = C^1_3$

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  - The third "roll" rotation, "rotates" the frame {3} basis vector to become the frame {4} basis vector $[\vec{x}\,^1_4, \vec{y}\,^1_4, \vec{z}\,^1_4] = R_{(\vec{x},\phi)}[\vec{x}\,^1_3, \vec{y}\,^1_3, \vec{z}\,^1_3] = C^1_4$

- In the case of fixed axis: Yaw (ψ), Pitch (θ), Roll (ϕ) rotation sequence

$$
\begin{aligned}
C_4^1 &= R_{(\vec{x},\phi)} R_{(\vec{y},\theta)} R_{(\vec{z},\psi)} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\theta c_\psi & -c_\theta s_\psi & s_\theta \\ c_\psi s_\theta s_\phi + c_\phi s_\psi & c_\phi c_\psi - s_\theta s_\phi s_\psi & -c_\theta s_\phi \\ s_\phi s_\psi - c_\phi c_\psi s_\theta & c_\psi s_\phi + c_\phi s_\theta s_\psi & c_\theta c_\phi \end{bmatrix}
\end{aligned} \tag{2}
$$

- Fixed Axis Rotations
  - Multiply on the LEFT
  - $R_{final} = R_n \dots R_2 R_1$

Fixed Axis Rotation

$$R_{resultant} = R_{fixed} R_{original}$$

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Fixed Axis Rotation

$$R_{resultant} = R_{fixed} R_{original}$$

- Relative (Euler) Axis Rotations
  - Multiply on the RIGHT
  - $R_{final} = R_1 R_2 \dots R_n$

Relative Axis Rotation

$$R_{resultant} = R_{original} R_{relative}$$

- Fixed Axis Rotations
  - Multiply on the LEFT
  - $R_{final} = R_n \dots R_2 R_1$

Fixed Axis Rotation

$R_{resultant} = R_{fixed} R_{original}$

- Relative (Euler) Axis Rotations
  - Multiply on the RIGHT
  - $R_{final} = R_1 R_2 \dots R_n$

Relative Axis Rotation

$R_{resultant} = R_{original} R_{relative}$

You can mix the two types of rotations