EE 570: Location and Navigation

Navigation Mathematics: Kinematics (Angle-Axis Rotation)

Stephen Bruder[1] Aly El-Osery[2]

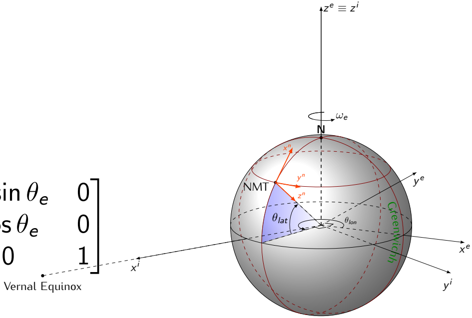
[1]Electrical and Computer Engineering Department, Embry-Riddle Aeronautical Univesity
Prescott, Arizona, USA
[2]Electrical Engineering Department, New Mexico Tech
Socorro, New Mexico, USA

January 28, 2014

- Recall: 3-parameter descriptions of rotation:
  - Fixed axis rotations,
  - Relative (or Euler) axis rotations, and
  - Angle-axis rotations
- for both fixed and relative axis format order/sequence is critical

- What is the description of the ECEF frame resolved in the ECI frame? (i.e. C_e^i)

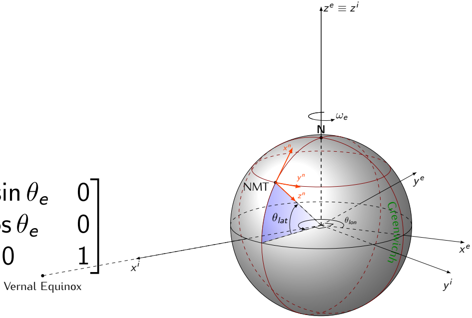
$$C_e^i = R_{(\vec{z},\theta_e)} = \begin{bmatrix} \cos\theta_e & -\sin\theta_e & 0 \\ \sin\theta_e & \cos\theta_e & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



- What is θ_e?

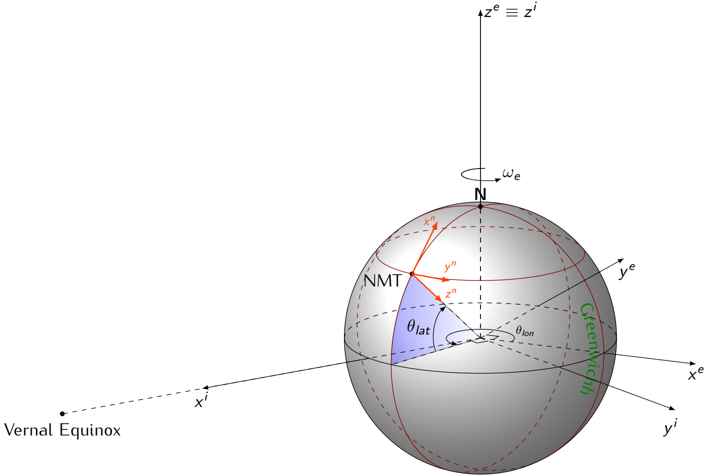
- What is the description of the ECEF frame resolved in the ECI frame? (i.e. C_e^i)

$$C_e^i = R_{(\vec{z},\theta_e)} = \begin{bmatrix} \cos\theta_e & -\sin\theta_e & 0 \\ \sin\theta_e & \cos\theta_e & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



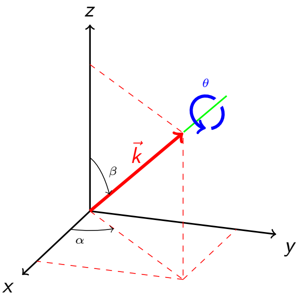
- What is θ_e?
- $\theta_e = \omega_{ie}(t - t_0)$

- What is the nav frame resolved in the ECEF frame (i.e. C_n^e)

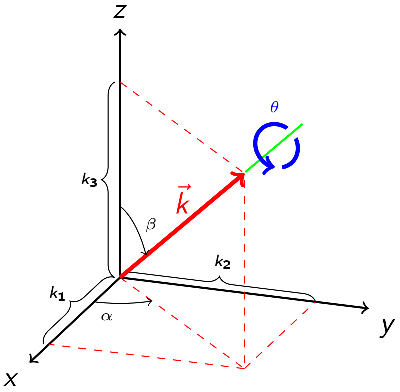


- The angle-axis format **does not** have the rotation in sequence issue
- **FACT**: Any rotation matrix C_b^a can be realized via rotating by an angle θ about an appropriately chosen axis of rotation ($\vec{k}$ say).

- Rotation matrix formed by rotating about some unit vector $\vec{k}$ by an angle of θ
  - Firstly, note that $\vec{k}$ need only stipulate a direction, and thus a unit-length vector is sufficient.
- This rotation matrix can be derived by rotating one of the principal axis (x, y, or z) onto the vector $\vec{k}$, then performing a rotation of θ, and finally undoing the original changes

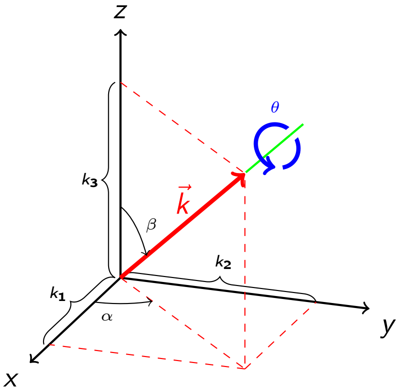


- By, noting that



- By, noting that

$$\sin\alpha = \frac{k_2}{\sqrt{k_1^2 + k_2^2}}$$
$$\cos\alpha = \frac{k_1}{\sqrt{k_1^2 + k_2^2}}$$
$$\sin\beta = \sqrt{k_1^2 + k_2^2}$$
$$\cos\beta = k_3$$



Angle-Axis

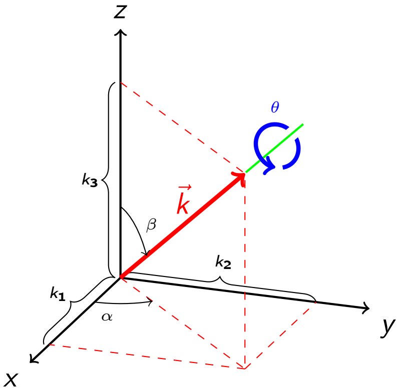
- By, noting that

$$\sin\alpha = \frac{k_2}{\sqrt{k_1^2 + k_2^2}}$$

$$\cos\alpha = \frac{k_1}{\sqrt{k_1^2 + k_2^2}}$$

$$\sin\beta = \sqrt{k_1^2 + k_2^2}$$

$$\cos\beta = k_3$$



- we can show that

$$R_{(\boldsymbol{k},\theta)} = \begin{bmatrix} k_1^2 V_\theta + c_\theta & k_1 k_2 V_\theta - k_3 s_\theta & k_1 k_3 V_\theta + k_2 s_\theta \\ k_1 k_2 V_\theta + k_3 s_\theta & k_2^2 V_\theta + c_\theta & k_2 k_3 V_\theta - k_1 s_\theta \\ k_1 k_3 V_\theta - k_2 s_\theta & k_2 k_3 V_\theta + k_1 s_\theta & k_3^2 V_\theta + c_\theta \end{bmatrix} \quad (1)$$

where $V_\theta \equiv 1 - \cos\theta$.

- Alternatively, the angle-axis rotation matrix is related to its equivalent angle-axis pair by

$$R_{(\vec{k},\theta(t))} = e^{\mathfrak{K}\theta(t)} \tag{2}$$

where

skew-symmetric

$$\mathfrak{K} = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix} \tag{3}$$

is the matrix version of the rotation axis vector $\vec{k} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}^T$.
Note: $\mathfrak{K}^T = -\mathfrak{K}$.

- Using Taylor expansion

$$R_{(\vec{k},\theta(t))} = e^{\mathfrak{K}\theta(t)} = \mathcal{I} + \mathfrak{K}\theta(t) + \frac{\mathfrak{K}^2\theta^2(t)}{2!} + \frac{\mathfrak{K}^3\theta^3(t)}{3!} + \cdots$$

- After a bit of manipulation we can show that

Rodrigues Formula

$$R_{(\vec{k},\theta(t))} = \mathcal{I} + \sin(\theta(t))\mathfrak{K} + [1 - \cos(\theta(t))]\,\mathfrak{K}^2 \quad (4)$$

- Multiplying out the *rhs* of the above equation gives us the same results as in Eq. 1

- Recall that

$$R_{(\mathbf{k},\theta)} = \begin{bmatrix} k_1^2 V_\theta + c_\theta & k_1 k_2 V_\theta - k_3 s_\theta & k_1 k_3 V_\theta + k_2 s_\theta \\ k_1 k_2 V_\theta + k_3 s_\theta & k_2^2 V_\theta + c_\theta & k_2 k_3 V_\theta - k_1 s_\theta \\ k_1 k_3 V_\theta - k_2 s_\theta & k_2 k_3 V_\theta + k_1 s_\theta & k_3^2 V_\theta + c_\theta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- What is the angle-axis pair $(\vec{k}, \theta)$ needed to realize this rotation matrix

- Recall that

$$R_{(\mathbf{k},\theta)} = \begin{bmatrix} k_1^2 V_\theta + c_\theta & k_1 k_2 V_\theta - k_3 s_\theta & k_1 k_3 V_\theta + k_2 s_\theta \\ k_1 k_2 V_\theta + k_3 s_\theta & k_2^2 V_\theta + c_\theta & k_2 k_3 V_\theta - k_1 s_\theta \\ k_1 k_3 V_\theta - k_2 s_\theta & k_2 k_3 V_\theta + k_1 s_\theta & k_3^2 V_\theta + c_\theta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- What is the angle-axis pair $(\vec{k}, \theta)$ needed to realize this rotation matrix
- Look at the trace of the matrix ($V_\theta \equiv 1 - \cos\theta$)

$$Tr(R) = \left[k_1^2 + k_2^2 + k_3^2\right](1 - \cos\theta) + 3\cos\theta = 1 + 2\cos\theta$$

$$\theta = \cos^{-1}\left(\frac{Tr(R) - 1}{2}\right) = \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right) \qquad (5)$$

- Also, looking for the axis of rotation

$$
\begin{aligned}
r_{32} - r_{23} &= 2k_1 s_\theta \\
r_{13} - r_{31} &= 2k_2 s_\theta \\
r_{21} - r_{12} &= 2k_3 s_\theta
\end{aligned}
$$

- Also, looking for the axis of rotation

$$r_{32} - r_{23} = 2k_1 s_\theta$$
$$r_{13} - r_{31} = 2k_2 s_\theta$$
$$r_{21} - r_{12} = 2k_3 s_\theta$$

- Therefore,

$$\vec{k} = \frac{1}{2s_\theta} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \tag{6}$$

- A satellite orbiting the earth can be made to point it's telescope at a desired star by performing
  1. Rotate about it's x-axis by $-30°$, then
  2. Rotate about it's new z-axis by $50°$, then finally
  3. Rotate about it's initial y-axis by $40°$.

- A satellite orbiting the earth can be made to point it's telescope at a desired star by performing
  1. Rotate about it's x-axis by $-30°$, then
  2. Rotate about it's new z-axis by $50°$, then finally
  3. Rotate about it's initial y-axis by $40°$.

- what is its final orientation *wrt* the starting orientation?

$$C_{final}^{start} = R_{(\vec{y},40°)} R_{(\vec{x},-30°)} R_{(\vec{z},50°)}$$

$$= \begin{bmatrix} 0.766044 & 0 & 0.642788 \\ 0 & 1 & 0 \\ -0.642788 & 0 & 0.766044 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.866025 & 0.5 \\ 0 & -0.5 & 0.866025 \end{bmatrix} \begin{bmatrix} 0.642788 & -0.766044 & 0 \\ 0.766044 & 0.642788 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.246202 & -0.793412 & 0.55667 \\ 0.663414 & 0.663414 & 0.5 \\ -0.706588 & 0.246202 & 0.246202 \end{bmatrix}$$

- In order to save enerty it is desirable to perform this change in orientation with only one rotation — How?

- In order to save enerty it is desirable to perform this change in orientation with only one rotation — How?
- Perform an angle-axis rotation

$$\theta = \cos^{-1}\left(\frac{Tr(R) - 1}{2}\right) = 76.5^\circ$$

$$\vec{k} = \frac{1}{2s_\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} = \begin{bmatrix} -0.130495 \\ 0.649529 \\ 0.749055 \end{bmatrix}$$