Lecture

Navigation Mathematics: Kinematics (Quaternions)

EE 570: Location and Navigation

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Singularity Problem

Euler angles and angle-axis rotation consist of only three elements and they are not unique, e.g., there are orientations that are represented by different Euler angles and angle-axis rotation.

Quaternion

Quaternions are 4-element representation of the rotation vectors. The additional element makes quaternions unique. Since quaternions are only 4 elements, they have the lowest dimentionality possible for a globally nonsignualar attitude representation.

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Quaternions

- Given a rotation vector defined as

$$\vec{K} = \theta\vec{k}$$

where $\vec{k}$ and θ are Euler axis and angle, respectively, a quaternion is defined as

$$\bar{q} = \begin{bmatrix} q_s \\ \vec{q} \end{bmatrix} = \begin{bmatrix} q_s \\ q_x \\ q_y \\ q_z \end{bmatrix} = \begin{bmatrix} \cos(\frac{\theta}{2}) \\ \vec{k}\sin(\frac{\theta}{2}) \end{bmatrix}$$

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Quaternion Properties

- Quaternion inverse

$$\bar{q}^{\,-1} = \begin{bmatrix} q_s \\ -q_x \\ -q_y \\ -q_z \end{bmatrix}$$

- Vector transformation Define

$$\breve{q} = \begin{bmatrix} 0 \\ \vec{q} \end{bmatrix}$$

then transforming a vector $\vec{v}^{\,p}$ defined in the *p*-frame may be transformed to the *i*-frame using

$$\breve{v}^i = \bar{q} \otimes \breve{v}^p \otimes \bar{q}^{\,-1}$$

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Quaternion Multiply

- Quaternion multiply

$$\bar{r} = \bar{q} \otimes \bar{p} = [\bar{q}\otimes]\bar{p} = \begin{bmatrix} q_s p_s - \vec{q} \cdot \vec{p} \\ q_s \vec{p} + p_s \vec{q} + \vec{q} \times \vec{p} \end{bmatrix}$$

  where

$$[\bar{q}\otimes] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & -q_z & q_y \\ q_y & q_z & q_s & -q_x \\ q_z & -q_y & q_x & q_s \end{bmatrix}$$

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Quaternion Multiply

- Quaternion multiply (corresponds to reverse order DCM)

$$\bar{r} = \bar{q} \circledast \bar{p} = [\bar{q}\circledast]\bar{p} = \begin{bmatrix} q_s p_s - \vec{q} \cdot \vec{p} \\ q_s \vec{p} + p_s \vec{q} - \vec{q} \times \vec{p} \end{bmatrix}$$

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$$\bar{q} \otimes \bar{p} = \bar{p} \circledast \bar{q}$$

  where

$$[\bar{q}\circledast] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & q_z & -q_y \\ q_y & -q_z & q_s & q_x \\ q_z & q_y & -q_x & q_s \end{bmatrix}$$

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Quaternion to DCM

$$\begin{aligned} \mathcal{T}(\bar{q}) &= (q_s^2 - |\vec{q}|^2)\mathcal{I} + 2q_s[\vec{q}\times] + 2\vec{q}\vec{q}^T = \\ &= \begin{bmatrix} q_s^2 + q_x^2 - q_y^2 - q_z^2 & 2(q_x q_y - q_s q_z) & 2(q_x q_z + q_s q_y) \\ 2(q_x q_y + q_s q_z) & q_s^2 - q_x^2 + q_y^2 - q_z^2 & 2(q_y q_z - q_s q_x) \\ 2(q_x q_z - q_s q_y) & 2(q_y q_z + q_s q_x) & q_s^2 - q_x^2 - q_y^2 + q_z^2 \end{bmatrix} \end{aligned}$$

The above DCM describes the orientation of the object frame as seen by the reference frame, e.g., (C_b^i)

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Quaternion to DCM

$$\begin{aligned} \mathcal{T}^T(\bar{q}) &= (q_s^2 - |\vec{q}|^2)\mathcal{I} - 2q_s[\vec{q}\times] + 2\vec{q}\vec{q}^T = \\ &= \begin{bmatrix} q_s^2 + q_x^2 - q_y^2 - q_z^2 & 2(q_x q_y + q_s q_z) & 2(q_x q_z - q_s q_y) \\ 2(q_x q_y - q_s q_z) & q_s^2 - q_x^2 + q_y^2 - q_z^2 & 2(q_y q_z + q_s q_x) \\ 2(q_x q_z + q_s q_y) & 2(q_y q_z - q_s q_x) & q_s^2 - q_x^2 - q_y^2 + q_z^2 \end{bmatrix} \end{aligned}$$

The above DCM describes the orientation of the reference frame as seen by the object frame, e.g., (C_i^b)

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Quaternion Identities

$$
\begin{aligned}
[\bar{q}^{\,-1}\otimes] &= [\bar{q}\otimes]^{-1} = [\bar{q}\otimes]^T \\
[\bar{q}^{\,-1}\circledast] &= [\bar{q}\circledast]^{-1} = [\bar{q}\circledast]^T \\
[\bar{q}\otimes] = e^{\frac{1}{2}[\breve{k}\otimes]} &= \cos(\theta/2)\mathcal{I} + \frac{1}{2}[\breve{k}\otimes]\frac{\sin(\theta/2)}{\theta/2} \\
[\bar{q}\circledast] = e^{\frac{1}{2}[\breve{k}\circledast]} &= \cos(\theta/2)\mathcal{I} + \frac{1}{2}[\breve{k}\circledast]\frac{\sin(\theta/2)}{\theta/2} \\
[\bar{q}\otimes][\bar{q}\circledast]^{-1} = [\bar{q}\circledast]^{-1}[\bar{q}\otimes] &= \begin{bmatrix} 1 & 0 \\ 0 & \mathcal{T}(\bar{q}) \end{bmatrix}
\end{aligned}
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Quaternion Identities

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\begin{aligned}
\bar{q} \otimes \bar{p} \otimes \bar{r} = (\bar{q} \otimes \bar{p}) \otimes \bar{r} &= \bar{q} \otimes (\bar{p} \otimes \bar{r}) \\
\bar{q} \circledast \bar{p} \circledast \bar{r} = (\bar{q} \circledast \bar{p}) \circledast \bar{r} &= \bar{q} \circledast (\bar{p} \circledast \bar{r}) \\
(\bar{q} \circledast \bar{p}) \otimes \bar{r} &\neq \bar{q} \circledast (\bar{p} \otimes \bar{r}) \\
(\bar{q} \otimes \bar{p}) \circledast \bar{r} &\neq \bar{q} \otimes (\bar{p} \circledast \bar{r})
\end{aligned}
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