Lecture

Navigation Mathematics: Kinematics (Angular Velocity)

EE 570: Location and Navigation

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Angular Velocity

- Since the relative and fixed axis rotations must be performed in a particular order, their derivatives are somewhat challenging
- The angle-axis format, however, is readily differentiable as we can encode the 3 parameters by

$$\vec{K} \equiv \vec{k}(t)\theta(t) = \begin{bmatrix} K_1(t) \\ K_2(t) \\ K_3(t) \end{bmatrix} \tag{1}$$

  where $\theta = \|\vec{K}\|$
- Hence,

$$\frac{d}{dt}\vec{K}(t) = \begin{bmatrix} \dot{K}_1(t) \\ \dot{K}_2(t) \\ \dot{K}_3(t) \end{bmatrix}$$

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Angular Velocity

- For a sufficiently "small" time interval we can often consider the axis of rotation to be $\approx$ constant (i.e.,$\vec{K}(t) = \vec{k}$)

$$\begin{aligned} \frac{d}{dt}\vec{K}(t) &= \frac{d}{dt}\left(\vec{k}\theta(t)\right) \\ &= \vec{k}\dot{\theta}(t) \end{aligned}$$

- This is referred to as the angular velocity ($\vec{\omega}(t)$) or the so called "body reference" angular velocity

  Angular Velocity

$$\vec{\omega}(t) \equiv \vec{k}\dot{\theta}(t) \tag{2}$$

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Angular Velocity

- This definition of the angular velocity can also be related back to the rotation matrix. Recalling that

$$C_b^a(t) = R_{(\vec{k}^{\,a}_{ab},\theta(t))} = e^{\mathfrak{K}^a_{ab}\theta(t)}$$

- Hence,

$$\begin{aligned}\frac{d}{dt}C_b^a(t) &= \frac{d}{dt}e^{\mathfrak{K}^a_{ab}\theta(t)} \\ &= \frac{\partial C_b^a}{\partial\theta}\frac{d\theta}{dt} \\ &= \mathfrak{K}^a_{ab}e^{\mathfrak{K}^a_{ab}\theta(t)}\dot{\theta}(t) \\ &= \left(\mathfrak{K}^a_{ab}\dot{\theta}(t)\right)C_b^a(t)\end{aligned}$$

$$\Rightarrow \dot{C}_b^a(t)\left[C_b^a(t)\right]^T = \mathfrak{K}^a_{ab}\dot{\theta}(t)$$

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Angular Velocity

- Notice that

$$\begin{aligned}\mathfrak{K}^a_{ab}\dot{\theta}(t) &= Skew\left[k^a_{ab}\right]\dot{\theta}(t) \\ &= Skew\left[k^a_{ab}\dot{\theta}(t)\right] \\ &= Skew\left[\omega^a_{ab}\right] = \Omega^a_{ab}\end{aligned}$$

- Therefore,

$$\dot{C}_b^a(t)\left[C_b^a(t)\right]^T = \Omega^a_{ab}$$

- or

$$\dot{C}_b^a = \Omega^a_{ab}C_b^a \tag{3}$$

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Properties of Skew-Symmetric Matrices

- $$\mathfrak{K}\vec{a} = \begin{bmatrix}0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0\end{bmatrix}\begin{bmatrix}a_1 \\ a_2 \\ a_3\end{bmatrix} = \begin{bmatrix}k_2a_3 - k_3a_2 \\ k_3a_1 - k_1a_3 \\ k_1a_2 - k_2a_1\end{bmatrix} = \vec{k}\times\vec{a}$$

- Hence, we can think of the skew-symmetric matrix as

$$\mathfrak{K} = [\vec{k}\times]$$

- or, in the case of angular velocity

$$\Omega = [\vec{\omega}\times]$$

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Properties of Skew-Symmetric Matrices

- $$\begin{aligned} C\Omega C^T\vec{b} &= C\left[\vec{\omega}\times\left(C^T\vec{b}\right)\right] \\ &= C\vec{\omega}\times\left(CC^T\vec{b}\right) \\ &= C\vec{\omega}\times\vec{b} \\ &= [C\vec{\omega}\times]\vec{b} \end{aligned}$$
- Therefore, $$C[\vec{\omega}\times]C^T = [C\vec{\omega}\times]$$
- or $$C[\vec{\omega}\times] = [C\vec{\omega}\times]C$$

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Properties of Skew-Symmetric Matrices

$$\begin{aligned} \dot{C}^a_b &= \Omega^a_{ab}C^a_b \\ &= [\vec{\omega}^{\,a}_{ab}\times]C^a_b \\ &= [C^a_b\vec{\omega}^{\,b}_{ab}\times]C^a_b \\ &= C^a_b[\vec{\omega}^{\,b}_{ab}\times] \\ &= C^a_b\Omega^b_{ab} \end{aligned}$$

$$\dot{C}^a_b = \Omega^a_{ab}C^a_b = C^a_b\Omega^b_{ab} \tag{4}$$

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Summary

Angular velocity can be

- described as a vector
  - the angular velocity of the b-frame *wrt* the a-frame resolved in the c-frame, $\vec{\omega}^{\,c}_{ab}$
  - $\vec{\omega}_{ab} = -\vec{\omega}_{ba}$
- described as a skew-symmetric matrix $\Omega^c_{ab} = [\vec{\omega}^{\,c}_{ab}\times]$
  - the skew-symmetric matrix is equivalent to the vector cross product
    - when pre-multiplying another vector
- related to the differential of the rotation matrix

$$\begin{aligned} \dot{C}^a_b &= \Omega^a_{ab}C^a_b = C^a_b\Omega^b_{ab} \\ \dot{C}^a_b &= -\Omega^a_{ba}C^a_b = -C^a_b\Omega^b_{ba} \end{aligned}$$

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Propagation of Angular Velocity

- Consider, $C^0_2 = C^0_1C^1_2$

$$\begin{aligned} \frac{d}{dt}C^0_2 &= \frac{d}{dt}C^0_1C^1_2 \\ \dot{C}^0_2 &= \dot{C}^0_1C^1_2 + C^0_1\dot{C}^1_2 \\ \Omega^0_{02}C^0_2 &= \Omega^0_{01}C^0_1C^1_2 + C^0_1C^1_2\Omega^2_{12} \\ \Omega^0_{02} &= \Omega^0_{01}C^0_2\left[C^0_2\right]^T + C^0_2\Omega^2_{12}\left[C^0_2\right]^T \\ [\omega^0_{02}\times] &= [\omega^0_{01}\times] + [C^0_2\omega^2_{12}\times] \\ \omega^0_{02} &= \omega^0_{01} + \omega^0_{12} \end{aligned}$$

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