EE 570: Location and Navigation

Navigation Mathematics: Kinematics (Angular Velocity)

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- Since the relative and fixed axis rotations must be performed in a particular order, their derivatives are somewhat challenging
- The angle-axis format, however, is readily differentiable as we can encode the 3 parameters by

$$\vec{K} \equiv \vec{k}(t)\theta(t) = \begin{bmatrix} K_1(t) \\ K_2(t) \\ K_3(t) \end{bmatrix} \tag{1}$$

where $\theta = \|\vec{K}\|$

- Hence,

$$\frac{d}{dt}\vec{K}(t) = \begin{bmatrix} \dot{K}_1(t) \\ \dot{K}_2(t) \\ \dot{K}_3(t) \end{bmatrix}$$

- For a sufficiently "small" time interval we can often consider the axis of rotation to be ≈ constant (i.e.,$\vec{K}(t) = \vec{k}$)

$$\frac{d}{dt}\vec{K}(t) = \frac{d}{dt}\left(\vec{k}\theta(t)\right) = \vec{k}\dot{\theta}(t)$$

- This is referred to as the angular velocity ($\vec{\omega}(t)$) or the so called "body reference" angular velocity

Angular Velocity

$$\vec{\omega}(t) \equiv \vec{k}\dot{\theta}(t) \tag{2}$$

- This definition of the angular velocity can also be related back to the rotation matrix. Recalling that

$$C_b^a(t) = R_{(\vec{k}^{\,a}_{ab}, \theta(t))} = e^{\mathfrak{K}^a_{ab}\theta(t)}$$

- Hence,

$$\begin{aligned}
\frac{d}{dt} C_b^a(t) &= \frac{d}{dt} e^{\mathfrak{K}^a_{ab}\theta(t)} \\
&= \frac{\partial C_b^a}{\partial \theta} \frac{d\theta}{dt} \\
&= \mathfrak{K}^a_{ab} e^{\mathfrak{K}^a_{ab}\theta(t)} \dot{\theta}(t) \\
&= \left(\mathfrak{K}^a_{ab} \dot{\theta}(t) \right) C_b^a(t)
\end{aligned}$$

$$\Rightarrow \dot{C}_b^a(t) \left[C_b^a(t) \right]^T = \mathfrak{K}^a_{ab} \dot{\theta}(t)$$

- Notice that

$$\mathfrak{K}^a_{ab}\dot{\theta}(t) = Skew\,[k^a_{ab}]\,\dot{\theta}(t)$$
$$= Skew\left[k^a_{ab}\dot{\theta}(t)\right]$$
$$= Skew\,[\vec{\omega}\,^a_{ab}] = \Omega^a_{ab}$$

- Therefore,

$$\dot{C}^a_b(t)\,[C^a_b(t)]^T = \Omega^a_{ab}$$

- or

$$\dot{C}^a_b = \Omega^a_{ab} C^a_b \tag{3}$$

- $$\mathfrak{K}\vec{a} = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} k_2 a_3 - k_3 a_2 \\ k_3 a_1 - k_1 a_3 \\ k_1 a_2 - k_2 a_1 \end{bmatrix} = \vec{k} \times \vec{a}$$
- Hence, we can think of the skew-symmetric matrix as

$$\mathfrak{K} = [\vec{k}\times]$$

- or, in the case of angular velocity

$$\Omega = [\vec{\omega}\times]$$

- $$\begin{aligned} C\Omega C^T\vec{b} &= C\left[\vec{\omega}\times\left(C^T\vec{b}\right)\right] \\ &= C\vec{\omega}\times\left(CC^T\vec{b}\right) \\ &= C\vec{\omega}\times\vec{b} \\ &= [C\vec{\omega}\times]\vec{b} \end{aligned}$$

- Therefore,

$$C[\vec{\omega}\times]C^T = [C\vec{\omega}\times]$$

- or

$$C[\vec{\omega}\times] = [C\vec{\omega}\times]C$$

$$
\begin{aligned}
\dot{C}_b^a &= \Omega_{ab}^a C_b^a \\
&= [\vec{\omega}\,_{ab}^a \times] C_b^a \\
&= [C_b^a \vec{\omega}\,_{ab}^b \times] C_b^a \\
&= C_b^a [\vec{\omega}\,_{ab}^b \times] \\
&= C_b^a \Omega_{ab}^b
\end{aligned}
$$

$$
\dot{C}_b^a = \Omega_{ab}^a C_b^a = C_b^a \Omega_{ab}^b \tag{4}
$$

Angular velocity can be

- described as a vector
  - the angular velocity of the b-frame *wrt* the a-frame resolved in the c-frame, $\vec{\omega}\,^{c}_{ab}$
  - $\vec{\omega}_{ab} = -\vec{\omega}_{ba}$
- described as a skew-symmetric matrix $\Omega^{c}_{ab} = [\vec{\omega}\,^{c}_{ab}\times]$
  - the skew-symmetric matrix is equivalent to the vector cross product
    - when pre-multiplying another vector
- related to the differential of the rotation matrix

$$\dot{C}^{a}_{b} = \Omega^{a}_{ab} C^{a}_{b} = C^{a}_{b} \Omega^{b}_{ab}$$
$$\dot{C}^{a}_{b} = -\Omega^{a}_{ba} C^{a}_{b} = -C^{a}_{b} \Omega^{b}_{ba}$$

- Consider, $C_2^0 = C_1^0 C_2^1$

$$\frac{d}{dt} C_2^0 = \frac{d}{dt} C_1^0 C_2^1$$
$$\dot{C}_2^0 = \dot{C}_1^0 C_2^1 + C_1^0 \dot{C}_2^1$$
$$\Omega_{02}^0 C_2^0 = \Omega_{01}^0 C_1^0 C_2^1 + C_1^0 C_2^1 \Omega_{12}^2$$
$$\Omega_{02}^0 = \Omega_{01}^0 C_2^0 \left[C_2^0\right]^T + C_2^0 \Omega_{12}^2 \left[C_2^0\right]^T$$
$$[\vec{\omega}\,_{02}^0 \times] = [\vec{\omega}\,_{01}^0 \times] + [C_2^0 \vec{\omega}\,_{12}^2 \times]$$
$$\vec{\omega}\,_{02}^0 = \vec{\omega}\,_{01}^0 + \vec{\omega}\,_{12}^0$$