EE 570: Location and Navigation

Navigation Mathematics: Kinematics (Linear Velocity)

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- Consider the motion of a fixed point (origin of frame {2}) in a rotating frame (frame {1}) as seen from an inertial (frame {0})
  - Frame 0 and 1 have the same origin
  - Frame 1 rotates (about a unit vector $\vec{k}$) *wrt* frame 0
  - The origin of frame 2 is fixed *wrt* frame 1
- Position:

$$\begin{aligned}\vec{r}^{\,0}_{02}(t) &= \underbrace{\vec{r}^{\,0}_{01}(t)}_{\vec{0}} + \vec{r}^{\,0}_{12}(t)\\ &= C^0_1(t)\vec{r}^{\,1}_{12}\end{aligned}$$

- Velocity

$$\begin{aligned}\dot{\vec{r}}^{\,0}_{02}(t) &= \frac{d}{dt} C^0_1(t)\vec{r}^{\,1}_{12} \\ &= \dot{C}^0_1(t)\vec{r}^{\,1}_{12} \\ &= [\vec{\omega}^{\,0}_{01}\times]C^0_1(t)\vec{r}^{\,1}_{12} \\ &= \vec{\omega}^{\,0}_{01} \times \vec{r}^{\,0}_{12}(t)\end{aligned}$$

- Acceleration

$$\ddot{\vec{r}}\,^0_{02} = \frac{d}{dt}\vec{\omega}\,^0_{01} \times \left(C_1^0(t)\vec{r}\,^1_{12}\right)$$
$$= \dot{\vec{\omega}}\,^0_{01} \times \left(C_1^0(t)\vec{r}\,^1_{12}\right) + \vec{\omega}\,^0_{01} \times \left(\dot{C}_1^0(t)\vec{r}\,^1_{12}\right)$$
$$= \dot{\vec{\omega}}\,^0_{01} \times \vec{r}\,^0_{12}(t) + \vec{\omega}\,^0_{01} \times \left(\vec{\omega}\,^0_{01} \times \vec{r}\,^0_{12}(t)\right)$$

Transverse accel

Centripetal accel ($\omega^2 r$)

- Consider the motion of a **moving** point (origin of frame {2}) in a rotating frame (frame {1}) as seen from an inertial (frame {0})
  - Frame 0 and 1 have the same origin
  - frame 1 rotates (about a unit vector $\vec{k}$) *wrt* frame 0
  - the origin of frame 2 is **moving** *wrt* frame 1
- Position:

$$\begin{aligned}\vec{r}^{\,0}_{02}(t) &= \cancelto{\vec{0}}{\vec{r}^{\,0}_{01}(t)} + \vec{r}^{\,0}_{12}(t) \\ &= C^0_1(t)\vec{r}^{\,1}_{12}(t)\end{aligned}$$

$\Rightarrow$ now a function of time !!

- Velocity

$$
\begin{aligned}
\dot{\vec{r}}^{\,0}_{02}(t) &= \frac{d}{dt} C^0_1(t)\vec{r}^{\,1}_{12} \\
&= \dot{C}^0_1(t)\vec{r}^{\,1}_{12} + C^0_1(t)\dot{\vec{r}}^{\,1}_{12}(t) \\
&= [\vec{\omega}^{\,0}_{01}\times]C^0_1(t)\vec{r}^{\,1}_{12} + C^0_1(t)\dot{\vec{r}}^{\,1}_{12}(t) \\
&= \vec{\omega}^{\,0}_{01} \times \vec{r}^{\,0}_{12}(t) + C^0_1(t)\dot{\vec{r}}^{\,1}_{12}(t)
\end{aligned}
$$

Note: $\dot{\vec{r}}^{\,0}_{02} \neq C^0_1(t)\dot{\vec{r}}^{\,1}_{12}(t)$

Linear Velocity

- Acceleration

$$
\begin{aligned}
\ddot{\vec{r}}^{\,0}_{02} &= \frac{d}{dt}\left\{\vec{\omega}^{\,0}_{01} \times \left(C^0_1(t)\vec{r}^{\,1}_{12}\right) + C^0_1\dot{\vec{r}}^{\,1}_{12}\right\} \\
&= \dot{\vec{\omega}}^{\,0}_{01} \times \left(C^0_1(t)\vec{r}^{\,1}_{12}\right) + \vec{\omega}^{\,0}_{01} \times \left(\dot{C}^0_1(t)\vec{r}^{\,1}_{12}\right) + \vec{\omega}^{\,0}_{01} \times \left(C^0_1\dot{\vec{r}}^{\,1}_{12}\right) + \dot{C}^0_1\dot{\vec{r}}^{\,1}_{12} + C^0_1\ddot{\vec{r}}^{\,1}_{12} \\
&= \dot{\vec{\omega}}^{\,0}_{01} \times \vec{r}^{\,0}_{12}(t) + \vec{\omega}^{\,0}_{01} \times \left(\vec{\omega}^{\,0}_{01} \times \vec{r}^{\,0}_{12}(t)\right) + 2\vec{\omega}^{\,0}_{01} \times \left(C^0_1\dot{\vec{r}}^{\,1}_{12}\right) + C^0_1\ddot{\vec{r}}^{\,1}_{12}
\end{aligned}
$$

Transverse accel

Centripetal accel ($\omega^2 r$)

Coriolis accel ($2\omega \times v$)

the accel of {2} org as seen in {1}, but resolved in {0}