

EE 570: Location and Navigation

Navigation Equations: ECI Mechanization

Stephen Bruder¹ Aly El-Osery²

¹Electrical and Computer Engineering Department, Embry-Riddle Aeronautical University
Prescott, Arizona, USA

²Electrical Engineering Department, New Mexico Tech
Socorro, New Mexico, USA

February 20, 2014

- Determine the position, velocity and attitude of the **body** frame *wrt* the **inertial** frame.
 - **Position** — Vector from the origin of the inertial frame to the origin of the body frame resolved in the inertial frame: \vec{r}_{ib}^i
 - **Velocity** — Velocity of the body frame *wrt* the inertial frame resolved in the inertial frame: \vec{v}_{ib}^i
 - **Attitude** — Orientation of the body frame *wrt* the inertial frame: C_b^i

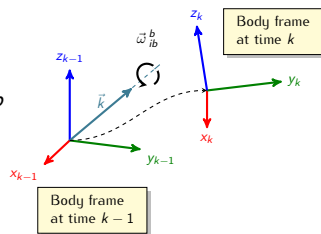
- Body orientation frame at time “ k ” wrt time “ $k - 1$ ”
 - $\Delta t = t_k - t_{k-1}$

$$\dot{C}_b^i = C_b^i \Omega_{ib}^b$$

$$= \lim_{\Delta t \rightarrow 0} \left(\frac{C_b^i(k) - C_b^i(k-1)}{\Delta t} \right) = C_b^i(k-1) \Omega_{ib}^b$$

$$C_b^i(+)-C_b^i(-) \approx C_b^i(-) \Omega_{ib}^b \Delta t$$

$$C_b^i(+)-C_b^i(-) \approx C_b^i(-) \left(\mathcal{I} + \Omega_{ib}^b \Delta t \right)$$



- Body orientation frame at time “ k ” wrt time “ $k - 1$ ”

- $\Delta t = t_k - t_{k-1}$

$$C_{b(k)}^i = C_{b(k-1)}^i C_{b(k)}^{b(k-1)}$$

$$\vec{\omega}_{ib}^b \Delta t = \vec{k} \Delta \theta$$

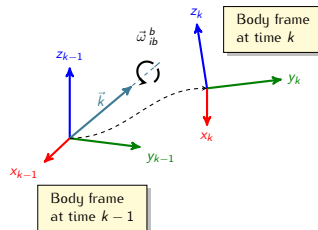
$$C_{b(k)}^{b(k-1)} = e^{\Omega_{ib}^b \Delta t} = e^{\mathfrak{K} \Delta \theta}$$

$$= \mathcal{I} + \mathfrak{K} \Delta \theta + \frac{\mathfrak{K}^2 \Delta \theta^2}{2!} + \frac{\mathfrak{K}^3 \Delta \theta^3}{3!} + \dots$$

$$= \mathcal{I} + \sin(\Delta \theta) \mathfrak{K} + [1 - \cos(\Delta \theta)] \mathfrak{K}^2$$

$$C_b^i(+)= C_b^i(-) C_{b(k)}^{b(k-1)}$$

$$\approx C_b^i(-) \left(\mathcal{I} + \Omega_{ib}^b \Delta t \right)$$



- Body orientation frame at time “ k ” wrt time “ $k - 1$ ”

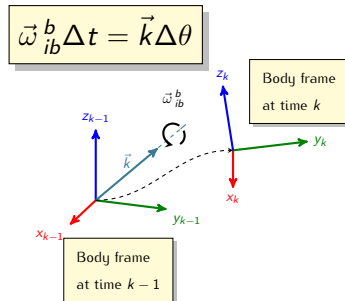
- $\Delta t = t_k - t_{k-1}$

$$\bar{q}_{b(k)}^i = \bar{q}_{b(k-1)}^i \otimes \bar{q}_{b(k)}^{b(k-1)}$$

$$\bar{q}_{b(k)}^{b(k-1)} = \begin{bmatrix} \cos\left(\frac{\Delta\theta}{2}\right) \\ \vec{k} \sin\left(\frac{\Delta\theta}{2}\right) \end{bmatrix}$$

$$\bar{q}_{b(+)}^i = \bar{q}_{b(-)}^i \otimes \bar{q}_{b(k)}^{b(k-1)}$$

Need to periodically renormalize \bar{q}



- High fidelity

$$\vec{\omega}_{ib}^b \Delta t = \vec{k} \Delta \theta$$

$$\mathfrak{K} = [\vec{k} \times]$$

$$C_b^i(+)=C_b^i(-)\left[\mathcal{I}+\sin(\Delta \theta) \mathfrak{K}+[1-\cos(\Delta \theta)] \mathfrak{K}^2\right] \quad (1)$$

or

$$\bar{q}_b^i(+)=\bar{q}_b^i(-) \otimes \begin{bmatrix} \cos\left(\frac{\Delta \theta}{2}\right) \\ \vec{k} \sin\left(\frac{\Delta \theta}{2}\right) \end{bmatrix} \quad (2)$$

- Low fidelity

$$C_b^i(+)\approx C_b^i(-)\left(\mathcal{I}+\Omega_{ib}^b \Delta t\right) \quad (3)$$

2 Specific force transformation

- Simply coordinatize the specific force

$$\vec{f}_{ib}^i = C_b^i(+)\vec{f}_{ib}^b \quad (4)$$

3 Velocity update

- Assuming that we are in space (i.e., no centrifugal component)

$$\vec{f}_{ib}^i = \vec{a}_{ib}^i - \vec{\gamma}_{ib}^i \quad \vec{a}_{ib}^i = \vec{f}_{ib}^i + \vec{\gamma}_{ib}^i \quad (5)$$

- Thus, by simple numerical integration

$$\vec{v}_{ib}^i(+)=\vec{v}_{ib}^i(-)+\vec{a}_{ib}^i\Delta t \quad (6)$$

4 Position update

- by simple numerical integration

$$\vec{r}_{ib}^i(+)=\vec{r}_{ib}^i(-)+\vec{v}_{ib}^i(-)\Delta t+\vec{a}_{ib}^i\frac{\Delta t^2}{2} \quad (7)$$

$$C_b^i(+)=C_b^i(-)\left[\mathcal{I}+\sin(\Delta\theta)\mathfrak{K}+[1-\cos(\Delta\theta)]\mathfrak{K}^2\right]$$

or

$$C_b^i(+)\approx C_b^i(-)\left(\mathcal{I}+\Omega_{ib}^b\Delta t\right)$$

or

$$\bar{q}_{ib}^i(+)=\bar{q}_{ib}^i(-)\otimes\begin{bmatrix}\cos\left(\frac{\Delta\theta}{2}\right) \\ \vec{k}\sin\left(\frac{\Delta\theta}{2}\right)\end{bmatrix}$$

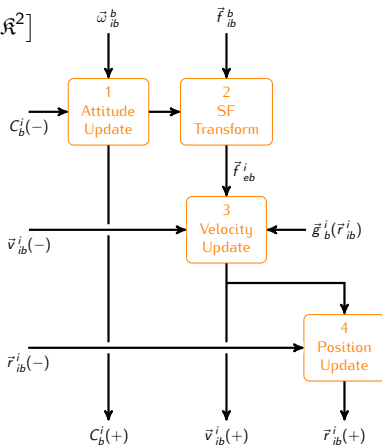
and

$$\vec{f}_{ib}^i=C_b^i(+)\vec{f}_{ib}^b$$

$$\vec{a}_{ib}^i=\vec{f}_{ib}^i+\vec{\gamma}_{ib}^i$$

$$\vec{v}_{ib}^i(+)=\vec{v}_{ib}^i(-)+\vec{a}_{ib}^i\Delta t$$

$$\vec{r}_{ib}^i(+)=\vec{r}_{ib}^i(-)+\vec{v}_{ib}^i(-)\Delta t+\vec{a}_{ib}^i\frac{\Delta t^2}{2}$$



- In continuous time notation
 - Attitude: $\dot{C}_b^i = C_b^i \Omega_{ib}^b$ or $\dot{\bar{q}}_b^i = \frac{1}{2}[\check{\omega}_{ib}^b \circledast] \bar{q}_b^i(t)$
 - Velocity: $\dot{\vec{r}}_{ib}^i = C_b^i \vec{f}_{ib}^b + \vec{\gamma}_{ib}^i$
 - Position: $\dot{\vec{r}}_{ib}^i = \vec{v}_{ib}^i$
- In State-space notation

$$\begin{bmatrix} \dot{\vec{r}}_{ib}^i \\ \dot{\vec{v}}_{ib}^i \\ \dot{C}_b^i \end{bmatrix} = \begin{bmatrix} \vec{v}_{ib}^i \\ C_b^i \vec{f}_{ib}^b + \vec{\gamma}_{ib}^i \\ C_b^i \Omega_{ib}^b \end{bmatrix} \quad (8)$$

or

$$\begin{bmatrix} \dot{\vec{r}}_{ib}^i \\ \dot{\vec{v}}_{ib}^i \\ \dot{\bar{q}}_b^i \end{bmatrix} = \begin{bmatrix} \vec{v}_{ib}^i \\ C_b^i \vec{f}_{ib}^b + \vec{\gamma}_{ib}^i \\ \frac{1}{2}[\check{\omega}_{ib}^b \circledast] \bar{q}_b^i(t) \end{bmatrix} \quad (9)$$

$$[\bar{q} \otimes] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & -q_z & q_y \\ q_y & q_z & q_s & -q_x \\ q_z & -q_y & q_x & q_s \end{bmatrix}$$

$$[\bar{q} \circledast] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & q_z & -q_y \\ q_y & -q_z & q_s & q_x \\ q_z & q_y & -q_x & q_s \end{bmatrix}$$