# Lecture

## Navigation Equations: ECEF Mechanization

### EE 570: Location and Navigation

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### ECEF Mechanization

- Determine the position, velocity and attitude of the **body** frame *wrt* the **Earth** frame.
  - **Position** — Vector from the origin of the earth frame to the origin of the body frame resolved in the earth frame: $\vec{r}^{\,e}_{eb}$
  - **Velocity** — Velocity of the body frame *wrt* the earth frame resolved in the earth frame: $\vec{v}^{\,e}_{eb}$
  - **Attitude** — Orientation of the body frame *wrt* the earth frame: $C^e_b$

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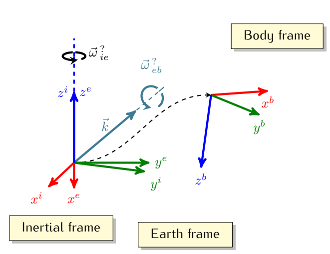
### Attitude — Method A

- Body orientation frame at time "$k$" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$
- Start with angular velocity

$$
\begin{aligned}
\vec{\omega}^{\,e}_{ib} &= \vec{\omega}^{\,e}_{ie} + \vec{\omega}^{\,e}_{eb}\\
\vec{\omega}^{\,e}_{eb} &= C^e_b \vec{\omega}^{\,b}_{ib} - \vec{\omega}^{\,e}_{ie}\\
\Omega^e_{eb} &= C^e_b \Omega^b_{ib} C^b_e - \Omega^e_{ie}
\end{aligned}
$$

$$C^e_b(+) - C^e_b(-) \approx \Delta t \Omega^e_{eb} C^e_b(-)$$

$$
\begin{aligned}
C^e_b(+) &\approx C^e_b(-) + \Delta t \left(C^e_b \Omega^b_{ib} C^b_e - \Omega^e_{ie}\right) C^e_b(-)\\
&= C^e_b(-)\left(\mathcal{I} + \Omega^b_{ib}\Delta t\right) - \Omega^e_{ie} C^e_b(-)\Delta t
\end{aligned}
$$

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### Attitude — Method B

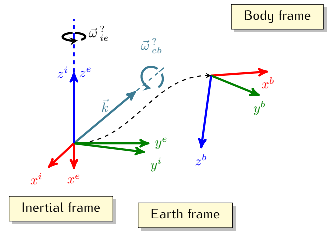
- Body orientation frame at time "$k$" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$
- Start with the angular velocity

$$\Omega^e_{eb} = C^e_b \Omega^b_{ib} C^b_e - \Omega^e_{ie}$$

$$C^e_b(+) = C^e_b(-) e^{\Omega^b_{eb}\Delta t} = e^{\Omega^e_{eb}\Delta t} C^e_b(-)$$

$$C^e_b(+) = \left[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1 - \cos(\Delta\theta)]\,\mathfrak{K}^2\right] C^e_b(-)$$

$$e^{\Omega^e_{eb}\Delta t} = e^{\mathfrak{K}\theta}$$

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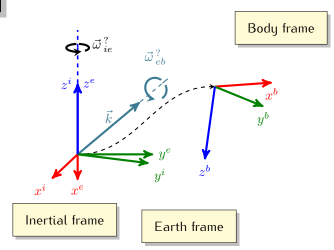
### Attitude — Method C

- Body orientation frame at time "$k$" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$

$$\vec{\omega}^{\,e}_{eb}\Delta t = \vec{k}\Delta\theta$$

$$\bar{q}^{\,e}_{b}(+) = \Delta\bar{q}^{\,e}_{b} \otimes \bar{q}^{\,e}_{b}(-)$$

$$\Delta\bar{q}^{\,e}_{b} = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix}$$




Need to periodically renormalize $\bar{q}$

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### Attitude Update— Summary

- High fidelity

$$C^e_b(+) = \left[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1 - \cos(\Delta\theta)]\,\mathfrak{K}^2\right] C^e_b(-) \tag{1}$$

or

$$\bar{q}^{\,e}_{b}(+) = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix} \otimes \bar{q}^{\,e}_{b}(-) \tag{2}$$

- Low fidelity

$$C^e_b(+) \approx C^e_b(-)\left(\mathcal{I} + \Omega^b_{ib}\Delta t\right) - \Omega^e_{ie} C^e_b(-)\Delta t \tag{3}$$

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### Steps 2–4

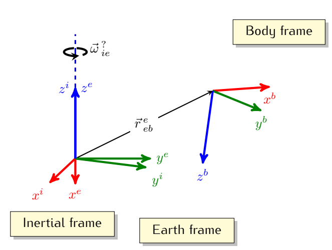
2. Specific force transformation
   - Simply coordinatize the specific force

$$\vec{f}^{\,e}_{ib} = C^e_b(+)\vec{f}^{\,b}_{ib} \tag{4}$$

3. Velocity update

$$\vec{r}^{\,i}_{ib} = \underbrace{\vec{r}^{\,i}_{ie}}_{0} + C^i_e \vec{r}^{\,e}_{eb} \Rightarrow \vec{r}^{\,e}_{eb} = C^e_i \vec{r}^{\,i}_{ib}$$

$$\begin{aligned} \vec{v}^{\,e}_{eb} &= \dot{\vec{r}}^{\,e}_{eb} \\ &= \dot{C}^e_i \vec{r}^{\,i}_{ib} + C^e_i \dot{\vec{r}}^{\,i}_{ib} \\ &= \Omega^e_{ei} C^e_i \vec{r}^{\,i}_{ib} + C^e_i \vec{v}^{\,i}_{ib} \\ &= -\Omega^e_{ie} \vec{r}^{\,e}_{eb} + C^e_i \vec{v}^{\,i}_{ib} \end{aligned}$$

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## Steps 2–4

$$\vec{a}^{\,e}_{eb} = \dot{\vec{v}}^{\,e}_{eb} = \frac{d}{dt}\left(-\Omega^e_{ie}\vec{r}^{\,e}_{eb} + C^e_i\vec{v}^{\,i}_{ib}\right)$$
$$= -\Omega^e_{ie}\dot{\vec{r}}^{\,e}_{eb} + \dot{C}^e_i\vec{v}^{\,i}_{ib} + C^e_i\dot{\vec{v}}^{\,i}_{ib}$$
$$= \; - \Omega^e_{ie}\vec{v}^{\,e}_{eb} + \Omega^e_{ei}\, C^e_i\vec{v}^{\,i}_{ib} + C^e_i\vec{a}^{\,i}_{ib}$$
$$= -\Omega^e_{ie}\vec{v}^{\,e}_{eb} - \Omega^e_{ie}\left[\vec{v}^{\,e}_{eb} + \Omega^e_{ie}\vec{r}^{\,e}_{eb}\right] + C^e_i\vec{a}^{\,i}_{ib}$$
$$= -2\Omega^e_{ie}\vec{v}^{\,e}_{eb} - \Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb} + \vec{a}^{\,e}_{ib}$$
$$= -2\Omega^e_{ie}\vec{v}^{\,e}_{eb} + \vec{f}^{\,e}_{ib} + \vec{g}^{\,e}_{b}$$

$C^e_i\vec{v}^{\,i}_{ib} = \Omega^e_{ie}\vec{r}^{\,e}_{eb} + \vec{v}^{\,e}_{eb}$

$\vec{f}^{\,?}_{ib} = \vec{a}^{\,?}_{ib} - \vec{\gamma}^{\,?}_{ib}$

$\vec{a}^{\,e}_{ib} = \vec{f}^{\,e}_{ib} + \vec{\gamma}^{\,e}_{ib}$

$\vec{g}^{\,e}_{b} = \vec{\gamma}^{\,e}_{ib} - \Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb}$

$\vec{a}^{\,e}_{ib} = \vec{f}^{\,e}_{ib} + \vec{g}^{\,e}_{b} + \Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb}$

$$\vec{v}^{\,e}_{eb}(+) = \vec{v}^{\,e}_{eb}(-) + \vec{a}^{\,e}_{eb}\Delta t$$
$$= \vec{v}^{\,e}_{eb}(-) + \left[\vec{f}^{\,e}_{ib} + \vec{g}^{\,e}_{b} - 2\Omega^i_{ie}\vec{v}^{\,e}_{eb}(-)\right]\Delta t \qquad (5)$$

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## Steps 2–4

4. Position update
   - by simple numerical integration

$$\vec{r}^{\,e}_{eb}(+) = \vec{r}^{\,e}_{eb}(-) + \vec{v}^{\,e}_{eb}(-)\Delta t + \vec{a}^{\,e}_{eb}\frac{\Delta t^2}{2} \qquad (6)$$

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## ECEF Mechanization Summary

$$C^e_b(+) = \left[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1-\cos(\Delta\theta)]\,\mathfrak{K}^2\right]C^e_b(-)$$

or

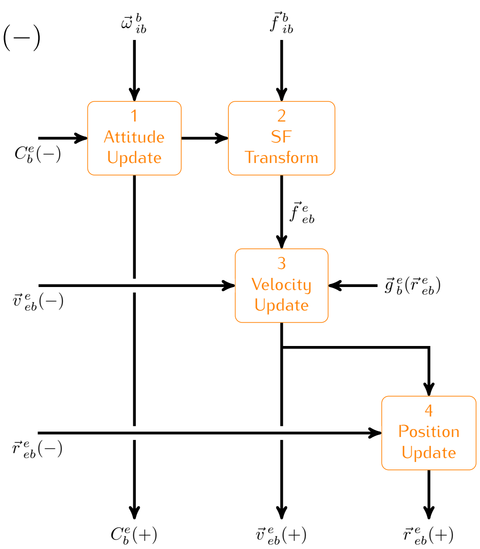
$$C^e_b(+) \approx C^e_b(-)\left(\mathcal{I} + \Omega^b_{ib}\Delta t\right) - \Omega^i_{ie}C^e_b(-)\Delta t$$

or

$$\bar{q}^{\,e}_{b}(+) = \begin{bmatrix}\cos(\frac{\Delta\theta}{2})\\ \vec{k}\sin(\frac{\Delta\theta}{2})\end{bmatrix} \otimes \bar{q}^{\,e}_{b}(-)$$

and

$$\vec{f}^{\,e}_{ib} = C^e_b(+)\vec{f}^{\,b}_{ib}$$
$$\vec{a}^{\,e}_{eb} = \vec{f}^{\,e}_{ib} + \vec{g}^{\,e}_{b} - 2\Omega^i_{ie}\vec{v}^{\,e}_{eb}(-)$$
$$\vec{v}^{\,e}_{eb}(+) = \vec{v}^{\,e}_{eb}(-) + \vec{a}^{\,e}_{eb}\Delta t$$
$$\vec{r}^{\,e}_{eb}(+) = \vec{r}^{\,e}_{eb}(-) + \vec{v}^{\,e}_{eb}(-)\Delta t + \vec{a}^{\,e}_{eb}\frac{\Delta t^2}{2}$$

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## ECEF Mechanization — Continuous Case

- In continuous time notation
  - Attitude: $\dot{C}^e_b = C^e_b\Omega^b_{eb}$ or $\dot{\bar{q}}^e_b = \frac{1}{2}[\breve{\omega}^b_{eb}\circledast]\bar{q}^{\,e}_{b}(t)$
  - Velocity: $\dot{\vec{v}}^{\,e}_{eb} = C^e_b\vec{f}^{\,b}_{ib} + \vec{g}^{\,e}_{b} - 2\Omega^i_{ie}\vec{v}^{\,e}_{eb}$
  - Position: $\dot{\vec{r}}^{\,e}_{ib} = \vec{v}^{\,e}_{eb}$

- In State-space notation

$$\begin{bmatrix} \dot{\vec{r}}^{\,e}_{eb} \\ \dot{\vec{v}}^{\,e}_{eb} \\ \dot{C}^i_b \end{bmatrix} = \begin{bmatrix} \vec{v}^{\,e}_{eb} \\ C^e_b \vec{f}^{\,b}_{ib} + \vec{g}^{\,e}_{b} - 2\Omega^i_{ie} \vec{v}^{\,e}_{eb} \\ C^e_b \Omega^b_{eb} \end{bmatrix} \tag{7}$$

or

$$\begin{bmatrix} \dot{\vec{r}}^{\,e}_{eb} \\ \dot{\vec{v}}^{\,e}_{eb} \\ \dot{\bar{q}}^e_b \end{bmatrix} = \begin{bmatrix} \vec{v}^{\,e}_{eb} \\ C^e_b \vec{f}^{\,b}_{ib} + \vec{g}^{\,e}_{b} - 2\Omega^i_{ie} \vec{v}^{\,e}_{eb} \\ \frac{1}{2}[\breve{\omega}^b_{eb} \circledast] \bar{q}^{\,e}_{b}(t) \end{bmatrix} \tag{8}$$

where $\vec{\omega}^{\,b}_{ib} = \vec{\omega}^{\,b}_{ie} + \vec{\omega}^{\,b}_{eb}$, and $\Omega^b_{eb} = \Omega^b_{ib} - \Omega^b_{ie}$. .11

## Appendix

$$[\bar{q}\otimes] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & -q_z & q_y \\ q_y & q_z & q_s & -q_x \\ q_z & -q_y & q_x & q_s \end{bmatrix}$$

$$[\bar{q}\circledast] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & q_z & -q_y \\ q_y & -q_z & q_s & q_x \\ q_z & q_y & -q_x & q_s \end{bmatrix}$$

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