EE 565: Position, Navigation and Timing

Error Mechanization (ECEF)

Aly El-Osery   Kevin Wedeward

Electrical Engineering Department, New Mexico Tech
Socorro, New Mexico, USA

In Collaboration with
Stephen Bruder
Electrical and Computer Engineering Department
Embry-Riddle Aeronautical Univesity
Prescott, Arizona, USA

April 2, 2018

Recall — ECEF Mechanization

- In continuous time notation
  - Attitude: $\dot{C}_b^e = C_b^e \Omega_{eb}^b$
  - Velocity: $\dot{\vec{v}}_{eb}^{\,e} = C_b^e \vec{f}_{ib}^{\,b} + \vec{g}_b^e - 2\Omega_{ie}^i \vec{v}_{eb}^{\,e}$
  - Position: $\dot{\vec{r}}_{ib}^{\,e} = \vec{v}_{eb}^{\,e}$
- In State-space notation

$$\vec{\omega}_{ib}^b = \vec{\omega}_{ie}^b + \vec{\omega}_{eb}^b$$

$$\Omega_{eb}^b = \Omega_{ib}^b - \Omega_{ie}^b$$

$$\begin{bmatrix} \dot{\vec{r}}_{eb}^{\,e} \\ \dot{\vec{v}}_{eb}^{\,e} \\ \dot{C}_b^i \end{bmatrix} = \begin{bmatrix} \vec{v}_{eb}^{\,e} \\ C_b^e \vec{f}_{ib}^{\,b} + \vec{g}_b^e - 2\Omega_{ie}^i \vec{v}_{eb}^{\,e} \\ C_b^e \Omega_{eb}^b \end{bmatrix} \tag{1}$$

- In continuous time notation
  - Attitude: $\dot{C}^e_b = C^e_b \Omega^b_{eb}$
  - Velocity: $\dot{\vec{v}}^{\,e}_{eb} = C^e_b \vec{f}^{\,b}_{ib} + \vec{g}^e_b - 2\Omega^i_{ie} \vec{v}^{\,e}_{eb}$
  - Position: $\dot{\vec{r}}^{\,e}_{ib} = \vec{v}^{\,e}_{eb}$
- In State-space notation

$$\vec{\omega}^b_{ib} = \vec{\omega}^b_{ie} + \vec{\omega}^b_{eb}$$

$$\Omega^b_{eb} = \Omega^b_{ib} - \Omega^b_{ie}$$

$$\begin{bmatrix} \dot{\vec{r}}^{\,e}_{eb} \\ \dot{\vec{v}}^{\,e}_{eb} \\ \dot{C}^i_b \end{bmatrix} = \begin{bmatrix} \vec{v}^{\,e}_{eb} \\ C^e_b \vec{f}^{\,b}_{ib} + \vec{g}^e_b - 2\Omega^i_{ie} \vec{v}^{\,e}_{eb} \\ C^e_b \Omega^b_{eb} \end{bmatrix} \tag{1}$$

Question

What is the effect of sensor noise in the measurements $\vec{\omega}^b_{ib}$ and $\vec{f}^b_{ib}$ on the navigation solution position, velocity and attitude?

$$\dot{C}_b^e = C_b^e \Omega_{eb}^b \; = C_b^e(\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e\right] =$$

$$\dot{C}_b^e = C_b^e \Omega_{eb}^b \; = C_b^e(\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e \Omega_{eb}^b = [\delta\dot{\vec{\psi}}_{eb}^e \times]\hat{C}_b^e + (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\dot{\hat{C}}_b^e =$$

$$\dot{C}_b^e = C_b^e \Omega_{eb}^b \; = C_b^e(\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e \Omega_{eb}^b = [\delta\dot{\vec{\psi}}_{eb}^e \times]\hat{C}_b^e + (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\dot{\hat{C}}_b^e =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e(\hat{\Omega}_{eb}^b + \delta\Omega_{ib}^b - \delta\Omega_{ie}^b) =$$

$$\dot{C}_b^e = C_b^e \Omega_{eb}^b \;= C_b^e(\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e \Omega_{eb}^b = [\delta\dot{\vec{\psi}}_{eb}^e \times]\hat{C}_b^e + (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\dot{\hat{C}}_b^e =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e(\hat{\Omega}_{eb}^b + \delta\Omega_{ib}^b - \delta\Omega_{ie}^b) =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e \hat{\Omega}_{eb}^b + \hat{C}_b^e(\delta\Omega_{ib}^b - \delta\Omega_{ie}^b) =$$

$$\because [\delta\vec{\psi}_{eb}^e \times]\delta\Omega_{eb}^b \approx 0$$

$$\dot{C}_b^e = C_b^e \Omega_{eb}^b = C_b^e(\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e \Omega_{eb}^b = [\delta\dot{\vec{\psi}}_{eb}^e \times]\hat{C}_b^e + (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\dot{\hat{C}}_b^e =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e(\hat{\Omega}_{eb}^b + \delta\Omega_{ib}^b - \delta\Omega_{ie}^b) =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e \hat{\Omega}_{eb}^b + \hat{C}_b^e(\delta\Omega_{ib}^b - \delta\Omega_{ie}^b) =$$

$$\dot{C}_b^e = C_b^e \Omega_{eb}^b \ = C_b^e(\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e \Omega_{eb}^b = [\delta\dot{\vec{\psi}}_{eb}^e \times]\hat{C}_b^e + (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\dot{\hat{C}}_b^e =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e(\hat{\Omega}_{eb}^b + \delta\Omega_{ib}^b - \delta\Omega_{ie}^b) =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e\hat{\Omega}_{eb}^b + \hat{C}_b^e(\delta\Omega_{ib}^b - \delta\Omega_{ie}^b) =$$

$$[\delta\dot{\vec{\psi}}_{eb}^e \times] = \hat{C}_b^e(\delta\Omega_{ib}^b - \delta\Omega_{ie}^b)\hat{C}_e^b = [\hat{C}_b^e(\delta\vec{\omega}_{ib}^b - \delta\vec{\omega}_{ie}^b)\times] \tag{2}$$

$$\dot{C}_b^e = C_b^e \Omega_{eb}^b = C_b^e(\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e \Omega_{eb}^b = [\delta\dot{\vec{\psi}}_{eb}^e \times]\hat{C}_b^e + (\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\dot{\hat{C}}_b^e =$$
$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e(\hat{\Omega}_{eb}^b + \delta\Omega_{ib}^b - \delta\Omega_{ie}^b) =$$
$$(\mathcal{I} + [\delta\vec{\psi}_{eb}^e \times])\hat{C}_b^e\hat{\Omega}_{eb}^b + \hat{C}_b^e(\delta\Omega_{ib}^b - \delta\Omega_{ie}^b) =$$

$$[\delta\dot{\vec{\psi}}_{eb}^e \times] = \hat{C}_b^e(\delta\Omega_{ib}^b - \delta\Omega_{ie}^b)\hat{C}_e^b = [\hat{C}_b^e(\delta\vec{\omega}_{ib}^b - \delta\vec{\omega}_{ie}^b)\times] \quad (2)$$

$$\delta\dot{\vec{\psi}}_{eb}^e = \hat{C}_b^e(\delta\omega_{ib}^b - \delta\vec{\omega}_{ie}^b) \quad (3)$$

ECEF Attitude Error (cont.)

$$
\begin{aligned}
\delta\dot{\vec{\psi}}^{e}_{eb} &= \hat{C}^{e}_{b}(\delta\vec{\omega}^{b}_{ib} - \delta\vec{\omega}^{b}_{ie}) \\
&= \hat{C}^{e}_{b}\delta\vec{\omega}^{b}_{ib} - \hat{C}^{e}_{b}(\vec{\omega}^{b}_{ie} - \hat{\vec{\omega}}^{b}_{ie}) \\
&= \hat{C}^{e}_{b}\delta\vec{\omega}^{b}_{ib} - (\hat{C}^{e}_{b}C^{b}_{e} - \mathcal{I})\vec{\omega}^{e}_{ie} \\
&= \hat{C}^{e}_{b}\delta\vec{\omega}^{b}_{ib} + \delta\vec{\psi}^{e}_{eb} \times \vec{\omega}^{e}_{ie}
\end{aligned}
$$

$$
\begin{aligned}
\delta\dot{\vec{\psi}}^e_{eb} &= \hat{C}^e_b(\delta\vec{\omega}^b_{ib} - \delta\vec{\omega}^b_{ie}) \\
&= \hat{C}^e_b\delta\vec{\omega}^b_{ib} - \hat{C}^e_b(\vec{\omega}^b_{ie} - \hat{\vec{\omega}}^b_{ie}) \\
&= \hat{C}^e_b\delta\vec{\omega}^b_{ib} - (\hat{C}^e_b C^b_e - \mathcal{I})\vec{\omega}^e_{ie} \\
&= \hat{C}^e_b\delta\vec{\omega}^b_{ib} + \delta\vec{\psi}^e_{eb} \times \vec{\omega}^e_{ie}
\end{aligned}
$$

$$
\delta\dot{\vec{\psi}}^e_{eb} = \hat{C}^e_b\delta\vec{\omega}^b_{ib} - \vec{\Omega}^e_{ie}\delta\vec{\psi}^e_{eb} \tag{4}
$$

$$\dot{\vec{v}}^{\,e}_{eb} = C^e_b \vec{f}^{\,b}_{ib} + \vec{g}^e_b - 2\Omega^e_{ie}\vec{v}^e_{eb} \quad (5)$$

$$\begin{aligned} \dot{\hat{\vec{v}}}^{\,e}_{eb} &= \hat{C}^e_b \hat{\vec{f}}^{\,b}_{ib} + \hat{\vec{g}}^e_b - 2\Omega^e_{ie}\hat{\vec{v}}^{\,e}_{eb} \\ &= (\mathcal{I} - [\delta\vec{\psi}^e_{eb}\times])C^e_b(\vec{f}^{\,b}_{ib} - \delta\vec{f}^{\,b}_{ib}) + \hat{\vec{g}}^e_b - 2\Omega^e_{ie}\hat{\vec{v}}^{\,e}_{eb} \end{aligned} \quad (6)$$

$$\dot{\vec{v}}^e_{eb} = C^e_b \vec{f}^b_{ib} + \vec{g}^e_b - 2\Omega^e_{ie}\vec{v}^e_{eb} \tag{5}$$

$$\begin{aligned}\dot{\hat{\vec{v}}}^e_{eb} &= \hat{C}^e_b \hat{\vec{f}}^b_{ib} + \hat{\vec{g}}^e_b - 2\Omega^e_{ie}\hat{\vec{v}}^e_{eb} \\ &= (\mathcal{I} - [\delta\vec{\psi}^e_{eb}\times])C^e_b(\vec{f}^b_{ib} - \delta\vec{f}^b_{ib}) + \hat{\vec{g}}^e_b - 2\Omega^e_{ie}\hat{\vec{v}}^e_{eb}\end{aligned} \tag{6}$$

$$\begin{aligned}\delta\dot{\vec{v}}^e_{eb} = \dot{\vec{v}}^e_{eb} - \dot{\hat{\vec{v}}}^e_{eb} &= [\delta\vec{\psi}^e_{eb}\times]C^e_b\vec{f}^b_{ib} + \hat{C}^e_b\delta\vec{f}^b_{ib} + \delta\vec{g}^e_b - 2\Omega^e_{ie}\delta\vec{v}^e_{eb} \\ &= [\delta\vec{\psi}^e_{eb}\times]\hat{C}^e_b\hat{\vec{f}}^b_{ib} + \hat{C}^e_b\delta\vec{f}^b_{ib} + \delta\vec{g}^e_b - 2\Omega^e_{ie}\delta\vec{v}^e_{eb}\end{aligned}$$

$$\dot{\vec{v}}^{e}_{eb} = C^e_b \vec{f}^{b}_{ib} + \vec{g}^e_b - 2\Omega^e_{ie}\vec{v}^e_{eb} \quad (5)$$

$$\begin{aligned}\dot{\hat{\vec{v}}}^{e}_{eb} &= \hat{C}^e_b \hat{\vec{f}}^{b}_{ib} + \hat{\vec{g}}^e_b - 2\Omega^e_{ie}\hat{\vec{v}}^{e}_{eb} \\ &= (\mathcal{I} - [\delta\vec{\psi}^e_{eb}\times])C^e_b(\vec{f}^{b}_{ib} - \delta\vec{f}^{b}_{ib}) + \hat{\vec{g}}^e_b - 2\Omega^e_{ie}\hat{\vec{v}}^{e}_{eb}\end{aligned} \quad (6)$$

$$\begin{aligned}\delta\dot{\vec{v}}^{e}_{eb} = \dot{\vec{v}}^{e}_{eb} - \dot{\hat{\vec{v}}}^{e}_{eb} &= [\delta\vec{\psi}^e_{eb}\times]C^e_b\vec{f}^{b}_{ib} + \hat{C}^e_b\delta\vec{f}^{b}_{ib} + \delta\vec{g}^e_b - 2\Omega^e_{ie}\delta\vec{v}^{e}_{eb} \\ &= [\delta\vec{\psi}^e_{eb}\times]\hat{C}^e_b\hat{\vec{f}}^{b}_{ib} + \hat{C}^e_b\delta\vec{f}^{b}_{ib} + \delta\vec{g}^e_b - 2\Omega^e_{ie}\delta\vec{v}^{e}_{eb}\end{aligned}$$

$$\delta\dot{\vec{v}}^{e}_{eb} = -[\hat{C}^e_b\hat{\vec{f}}^{b}_{ib}\times]\delta\vec{\psi}^e_{eb} + \hat{C}^e_b\delta\vec{f}^{b}_{ib} + \delta\vec{g}^e_b - 2\Omega^e_{ie}\hat{\vec{v}}^{e}_{eb} \quad (7)$$

Using Taylor series expansion, the gravity error as a function of position estimates and errors is derived to be

$$\delta\vec{g}_b^e \approx \frac{2g_0(\hat{L}_b)}{r_{eS}^e(\hat{L}_b)} \frac{\hat{\vec{r}}_{eb}^{\,e}}{|\hat{\vec{r}}_{eb}^{\,e}|^2} (\hat{\vec{r}}_{eb}^{\,e})^T \delta\vec{r}_{eb}^{\,e} \tag{8}$$

$$\dot{\vec{r}}^{e}_{eb} = \vec{v}^{e}_{eb} \tag{9}$$

$$\dot{\vec{r}}^{\,e}_{eb} = \vec{v}^{\,e}_{eb} \tag{9}$$

$$\delta\dot{\vec{r}}^{\,e}_{eb} = \delta\vec{v}^{\,e}_{eb} \tag{10}$$

$$\begin{pmatrix} \delta\dot{\vec{\psi}}^{e}_{eb} \\ \delta\dot{\vec{v}}^{\,e}_{eb} \\ \delta\dot{\vec{r}}^{\,e}_{eb} \end{pmatrix} = \begin{bmatrix} -\Omega^{e}_{ie} & 0_{3\times3} & 0_{3\times3} \\ -[\hat{C}^{e}_{b}\hat{\vec{f}}^{\,b}_{ib}\times] & -2\Omega^{e}_{ie} & \frac{2g_0(\hat{L}_b)}{r^{e}_{eS}(\hat{L}_b)}\frac{\hat{\vec{r}}^{e}_{eb}}{|\hat{\vec{r}}^{e}_{eb}|^2}(\hat{\vec{r}}^{e}_{eb})^T \\ 0_{3\times3} & \mathcal{I}_{3\times3} & 0_{3\times3} \end{bmatrix} \begin{pmatrix} \delta\vec{\psi}^{e}_{eb} \\ \delta\vec{v}^{\,e}_{eb} \\ \delta\vec{r}^{\,e}_{eb} \end{pmatrix} + \begin{bmatrix} 0 & \hat{C}^{e}_{b} \\ \hat{C}^{e}_{b} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \delta\vec{f}^{\,b}_{ib} \\ \delta\vec{\omega}^{b}_{ib} \end{pmatrix} \tag{11}$$

- Truth value

$$\vec{x}$$

- Measured value

$$\tilde{\vec{x}}$$

- Estimated or computed value

$$\hat{\vec{x}}$$

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value

$$\vec{x}$$

Nothing above

- Measured value

$$\tilde{\vec{x}}$$

- Estimated or computed value

$$\hat{\vec{x}}$$

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value

$$\vec{x}$$

- Measured value

$$\tilde{\vec{x}}$$

Use "tilde"

- Estimated or computed value

$$\hat{\vec{x}}$$

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

Notation Used

- Truth value

$$\vec{x}$$

- Measured value

$$\tilde{\vec{x}}$$

- Estimated or computed value

$$\hat{\vec{x}}$$

"Use hat"

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

Notation Used

- Truth value

$$\vec{x}$$

- Measured value

$$\tilde{\vec{x}}$$

- Estimated or computed value

$$\hat{\vec{x}}$$

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$
Let's assume we have an estimate of $\vec{x}$, i.e., $\hat{\vec{x}}$ such that $\vec{x} = \hat{\vec{x}} + \delta\vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \tag{12}$$

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$
Let's assume we have an estimate of $\vec{x}$, i.e., $\hat{\vec{x}}$ such that $\vec{x} = \hat{\vec{x}} + \delta\vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \tag{12}$$

Using Taylor series expansion

$$f(\hat{\vec{x}} + \delta\vec{x}, t) = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}}, t) + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T$$
$$\approx \dot{\hat{\vec{x}}} + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x}$$

Linearization using Taylor Series Expansion

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$
Let's assume we have an estimate of $\vec{x}$, i.e., $\hat{\vec{x}}$ such that $\vec{x} = \hat{\vec{x}} + \delta\vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \tag{12}$$

Using Taylor series expansion

$$f(\hat{\vec{x}} + \delta\vec{x}, t) = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}}, t) + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T$$

$$\approx \dot{\hat{\vec{x}}} + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x}$$

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$
Let's assume we have an estimate of $\vec{x}$, i.e., $\hat{\vec{x}}$ such that $\vec{x} = \hat{\vec{x}} + \delta\vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \tag{12}$$

Using Taylor series expansion

$$f(\hat{\vec{x}} + \delta\vec{x}, t) = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = \; f(\hat{\vec{x}}, t) \; + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T$$
$$\approx \; \dot{\hat{\vec{x}}} \; + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x}$$

$$\Rightarrow \delta\dot{\vec{x}} \approx \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \tag{13}$$

Initially the accelerometer and gyroscope measurements, $\tilde{\vec{f}}^{\,b}_{ib}$ and $\tilde{\vec{\omega}}^{b}_{ib}$, respectively, will be modeled as

$$\tilde{\vec{f}}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} + \Delta\vec{f}^{\,b}_{ib} \tag{14}$$

$$\tilde{\vec{\omega}}^{b}_{ib} = \vec{\omega}^{b}_{ib} + \Delta\vec{\omega}^{b}_{ib} \tag{15}$$

where $\vec{f}^{\,b}_{ib}$ and $\vec{\omega}^{b}_{ib}$ are the specific force and angular rates, respectively; and $\Delta\vec{f}^{\,b}_{ib}$ and $\Delta\vec{\omega}^{b}_{ib}$ represents the errors. In later lectures we will discuss more detailed description of these errors.

Initially the accelerometer and gyroscope measurements, $\tilde{\vec{f}}^{\,b}_{ib}$ and $\tilde{\vec{\omega}}^{b}_{ib}$, respectively, will be modeled as

$$\tilde{\vec{f}}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} + \Delta\vec{f}^{\,b}_{ib} \quad (14)$$

$$\tilde{\vec{\omega}}^{b}_{ib} = \vec{\omega}^{b}_{ib} + \Delta\vec{\omega}^{b}_{ib} \quad (15)$$

these terms may be expanded further

where $\vec{f}^{\,b}_{ib}$ and $\vec{\omega}^{b}_{ib}$ are the specific force and angular rates, respectively; and $\Delta\vec{f}^{\,b}_{ib}$ and $\Delta\vec{\omega}^{b}_{ib}$ represents the errors. In later lectures we will discuss more detailed description of these errors.

Accelerometers

$$\tilde{\vec{f}}^{\,b}_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}^{\,b}_{ib} + \vec{nl}_a + \vec{w}_a \tag{16}$$

Gyroscopes

$$\tilde{\vec{\omega}}^{b}_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}^{b}_{ib} + G_g\vec{f}^{\,b}_{ib} + \vec{w}_g \tag{17}$$

Accelerometers

$$\tilde{\vec{f}}\,^{b}_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}\,^{b}_{ib} + \vec{nl}_a + \vec{w}_a \tag{16}$$

Biases

Gyroscopes

$$\tilde{\vec{\omega}}^{b}_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}^{b}_{ib} + G_g\vec{f}\,^{b}_{ib} + \vec{w}_g \tag{17}$$

Accelerometers

$$\tilde{\vec{f}}\,^b_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}\,^b_{ib} + \vec{nl}_a + \vec{w}_a \quad (16)$$

Misalignment and SF Errors

Gyroscopes

$$\tilde{\vec{\omega}}^b_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}^b_{ib} + G_g\vec{f}\,^b_{ib} + \vec{w}_g \quad (17)$$

Error Modeling Example

Accelerometers

$$\tilde{\vec{f}}^{\,b}_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}^{\,b}_{ib} + \vec{nl}_a + \vec{w}_a \tag{16}$$

Non-linearity

Gyroscopes

$$\tilde{\vec{\omega}}^{b}_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}^{b}_{ib} + G_g\vec{f}^{\,b}_{ib} + \vec{w}_g \tag{17}$$

Accelerometers

$$\tilde{\vec{f}}^{\,b}_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}^{\,b}_{ib} + \vec{nl}_a + \vec{w}_a \tag{16}$$

G-Sensitivity

Gyroscopes

$$\tilde{\vec{\omega}}^{b}_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}^{b}_{ib} + G_g\vec{f}^{\,b}_{ib} + \vec{w}_g \tag{17}$$

Accelerometers

$$\tilde{\vec{f}}^{\,b}_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}^{\,b}_{ib} + \vec{nl}_a + \vec{w}_a \tag{16}$$

Noise

Gyroscopes

$$\tilde{\vec{\omega}}^{b}_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}^{b}_{ib} + G_g\vec{f}^{\,b}_{ib} + \vec{w}_g \tag{17}$$

Pos, Vel, Force and Angular Rate Errors

- Position error

$$\delta\vec{r}^{\gamma}_{\beta b} = \vec{r}^{\gamma}_{\beta b} - \hat{\vec{r}}^{\gamma}_{\beta b} \tag{18}$$

- Velocity error

$$\delta\vec{v}^{\gamma}_{\beta b} = \vec{v}^{\gamma}_{\beta b} - \hat{\vec{v}}^{\gamma}_{\beta b} \tag{19}$$

- Specific force errors

$$\delta\vec{f}^{b}_{ib} = \vec{f}^{b}_{ib} - \hat{\vec{f}}^{b}_{ib} \tag{20}$$

$$\Delta_e\vec{f}^{b}_{ib} = \Delta\vec{f}^{b}_{ib} - \Delta\hat{\vec{f}}^{b}_{ib} = -\delta\vec{f}^{b}_{ib} \tag{21}$$

- Angular rate errors

$$\delta\vec{\omega}^{b}_{ib} = \vec{\omega}^{b}_{ib} - \hat{\vec{\omega}}^{b}_{ib} \tag{22}$$

$$\Delta_e\vec{\omega}^{b}_{ib} = \Delta\vec{\omega}^{b}_{ib} - \Delta\hat{\vec{\omega}}^{b}_{ib} = -\delta\vec{\omega}^{b}_{ib} \tag{23}$$

Attitude Error Definition

Define

$$\delta C_b^\gamma = C_b^\gamma \hat{C}_\gamma^b = e^{[\delta\vec{\psi}^{\gamma}_{\gamma b}\times]} \approx \mathcal{I} + [\delta\vec{\psi}^{\gamma}_{\gamma b}\times] \tag{24}$$

This is the error in attitude resulting from errors in estimating the angular rates.

The attitude error is a multiplicative small angle transformation from the actual frame to the computed frame

$$\hat{C}_b^\gamma = (\mathcal{I} - [\delta\vec{\psi}_{\gamma b}^{\gamma}\times])C_b^\gamma \tag{25}$$

Similarly,

$$C_b^\gamma = (\mathcal{I} + [\delta\vec{\psi}_{\gamma b}^{\gamma}\times])\hat{C}_b^\gamma \tag{26}$$

Similarly we can attempt to estimate the specific force and angular rate by applying correction based on our estimate of the error.

$$\hat{\vec{f}}_{ib}^{\,b} = \tilde{\vec{f}}_{ib}^{\,b} - \Delta\hat{\vec{f}}_{ib}^{\,b} \tag{27}$$

$$\hat{\vec{\omega}}_{ib}^{b} = \tilde{\vec{\omega}}_{ib}^{b} - \Delta\hat{\vec{\omega}}_{ib}^{b} \tag{28}$$

where $\hat{\vec{f}}_{ib}^{\,b}$ and $\hat{\vec{\omega}}_{ib}^{b}$ are the accelerometer and gyroscope estimated calibration values, respectively.