EE 565: Position, Navigation and Timing

Error Mechanization (ECI)

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We have already derived the kinematic models in several frames. These models may be written in the form

$$\dot{\vec{x}} = \mathrm{f}(\vec{x}, \vec{u}) \tag{1}$$

where f is possibly non-linear.

Due to errors in the measurements we estimate $\vec{x}$ by integrating

$$\dot{\hat{\vec{x}}} = \mathrm{f}(\hat{\vec{x}}, \hat{\vec{u}}) \quad (2)$$

where $\hat{\vec{u}}$ is the measurement vector from the sensors after applying calibration corrections.

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If somehow we can model and possibly measure the error in the state we can then subtract it from the estimate to obtain an accurate position, velocity and attitude. We may also want to linearize the problem so that linear estimation approaches could be used.

All accelerometers and gyroscopes suffer from

- Biases
- Scale factor
- Cross-coupling
- Random noise

- *Fixed Errors*: deterministic and are present all the time, hence can be removed using calibration.
- *Temperature Dependent*: variations dependent on temperature and also may be modeled and characterized during calibration.
- *Run-to-run*: changes in the sensor error every time the sensor is run and is random in nature.
- *In-run*: random variations as the sensor is running.

- Truth value

$$\vec{x}$$

- Measured value

$$\tilde{\vec{x}}$$

- Estimated or computed value

$$\hat{\vec{x}}$$

- Error

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value

$$\vec{x}$$

Nothing above

- Measured value

$$\tilde{\vec{x}}$$

- Estimated or computed value

$$\hat{\vec{x}}$$

- Error

$$\delta \vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value
  $$\vec{x}$$
- Measured value
  $$\tilde{\vec{x}}$$
  Use "tilde"
- Estimated or computed value
  $$\hat{\vec{x}}$$
- Error
  $$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

- Truth value $\vec{x}$
- Measured value $\tilde{\vec{x}}$
- Estimated or computed value $\hat{\vec{x}}$ "Use hat"
- Error $\delta\vec{x} = \vec{x} - \hat{\vec{x}}$

- Truth value

$$\vec{x}$$

- Measured value

$$\tilde{\vec{x}}$$

- Estimated or computed value

$$\hat{\vec{x}}$$

- Error

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Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$

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Using Taylor series expansion

$$f(\hat{\vec{x}} + \delta\vec{x}, t) = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = \ f(\hat{\vec{x}}, t) \ + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T$$

$$\approx \ \dot{\hat{\vec{x}}} \ + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x}$$

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Using Taylor series expansion

$$\begin{aligned} f(\hat{\vec{x}} + \delta\vec{x}, t) = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} &= \ f(\hat{\vec{x}}, t) \ + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} + H.O.T \\ &\approx \ \dot{\hat{\vec{x}}} \ + \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \end{aligned}$$

$$\Rightarrow \delta\dot{\vec{x}} \approx \left.\frac{\partial f(\vec{x}, t)}{\partial \vec{x}}\right|_{\vec{x}=\hat{\vec{x}}} \delta\vec{x} \tag{4}$$

Initially the accelerometer and gyroscope measurements, $\tilde{\vec{f}}_{ib}^{\,b}$ and $\tilde{\vec{\omega}}_{ib}^{\,b}$, respectively, will be modeled as

$$\tilde{\vec{f}}_{ib}^{\,b} = \vec{f}_{ib}^{\,b} + \Delta\vec{f}_{ib}^{\,b} \tag{5}$$

$$\tilde{\vec{\omega}}_{ib}^{\,b} = \vec{\omega}_{ib}^{\,b} + \Delta\vec{\omega}_{ib}^{\,b} \tag{6}$$

where $\vec{f}_{ib}^{\,b}$ and $\vec{\omega}_{ib}^{\,b}$ are the specific force and angular rates, respectively; and $\Delta\vec{f}_{ib}^{\,b}$ and $\Delta\vec{\omega}_{ib}^{\,b}$ represents the errors. In later lectures we will discuss more detailed description of these errors.

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these terms may be expanded further

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Accelerometers

$$\tilde{\vec{f}}\,{}^{b}_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}\,{}^{b}_{ib} + \vec{nl}_a + \vec{w}_a \tag{7}$$

Gyroscopes

$$\tilde{\vec{\omega}}\,{}^{b}_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}\,{}^{b}_{ib} + G_g\vec{f}\,{}^{b}_{ib} + \vec{w}_g \tag{8}$$

Accelerometers

$$\tilde{\vec{f}}_{ib}^{b} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^{b} + \vec{nl}_a + \vec{w}_a \quad (7)$$

Biases

Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^{b} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^{b} + G_g\vec{f}_{ib}^{b} + \vec{w}_g \quad (8)$$

Accelerometers

$$\tilde{\vec{f}}\,^b_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}\,^b_{ib} + \vec{nl}_a + \vec{w}_a \tag{7}$$

Misalignment and SF Errors

Gyroscopes

$$\tilde{\vec{\omega}}\,^b_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}\,^b_{ib} + G_g\vec{f}\,^b_{ib} + \vec{w}_g \tag{8}$$

Accelerometers

$$\tilde{\vec{f}}^{\,b}_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}^{\,b}_{ib} + \vec{nl}_a + \vec{w}_a \tag{7}$$

Non-linearity

Gyroscopes

$$\tilde{\vec{\omega}}^{\,b}_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}^{\,b}_{ib} + G_g\vec{f}^{\,b}_{ib} + \vec{w}_g \tag{8}$$

Error Modeling Example

Accelerometers

$$\tilde{\vec{f}}^{\,b}_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}^{\,b}_{ib} + \vec{nl}_a + \vec{w}_a \tag{7}$$

G-Sensitivity

Gyroscopes

$$\tilde{\vec{\omega}}^{\,b}_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}^{\,b}_{ib} + G_g\vec{f}^{\,b}_{ib} + \vec{w}_g \tag{8}$$

Accelerometers

$$\tilde{\vec{f}}\,^b_{ib} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}\,^b_{ib} + \vec{nl}_a + \vec{w}_a \tag{7}$$

Noise

Gyroscopes

$$\tilde{\vec{\omega}}\,^b_{ib} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}\,^b_{ib} + G_g\vec{f}\,^b_{ib} + \vec{w}_g \tag{8}$$

Define the error state vector as

$$\delta\vec{x}^{\gamma}_{INS} = \begin{pmatrix} \delta\vec{\psi}^{\gamma}_{\gamma b} \\ \delta\vec{v}^{\gamma}_{\beta b} \\ \delta\vec{r}^{\gamma}_{\beta b} \end{pmatrix}, \quad \gamma, \beta \in i, e, n \tag{9}$$

Think of $\delta\vec{x}$ as the truth minus the estimate, i.e.,

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}} \tag{10}$$

The subtraction doesn't apply to the attitude component of the vector and needs to be treated differently

- Position error

$$\delta\vec{r}^{\,\gamma}_{\beta b} = \vec{r}^{\,\gamma}_{\beta b} - \hat{\vec{r}}^{\,\gamma}_{\beta b} \tag{11}$$

- Velocity error

$$\delta\vec{v}^{\,\gamma}_{\beta b} = \vec{v}^{\,\gamma}_{\beta b} - \hat{\vec{v}}^{\,\gamma}_{\beta b} \tag{12}$$

- Specific force errors

$$\delta\vec{f}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} - \hat{\vec{f}}^{\,b}_{ib} \tag{13}$$

$$\Delta_e\vec{f}^{\,b}_{ib} = \Delta\vec{f}^{\,b}_{ib} - \Delta\hat{\vec{f}}^{\,b}_{ib} = -\delta\vec{f}^{\,b}_{ib} \tag{14}$$

- Angular rate errors

$$\delta\vec{\omega}^{\,b}_{ib} = \vec{\omega}^{\,b}_{ib} - \hat{\vec{\omega}}^{\,b}_{ib} \tag{15}$$

$$\Delta_e\vec{\omega}^{\,b}_{ib} = \Delta\vec{\omega}^{\,b}_{ib} - \Delta\hat{\vec{\omega}}^{\,b}_{ib} = -\delta\vec{\omega}^{\,b}_{ib} \tag{16}$$

Define

$$\delta C_b^\gamma = C_b^\gamma \hat{C}_\gamma^b = e^{[\delta\vec{\psi}^{\gamma}_{\gamma b}\times]} \approx \mathcal{I} + [\delta\vec{\psi}^{\gamma}_{\gamma b}\times] \tag{17}$$

This is the error in attitude resulting from errors in estimating the angular rates.

The attitude error is a multiplicative small angle transformation from the actual frame to the computed frame

$$\hat{C}_b^\gamma = (\mathcal{I} - [\delta\vec{\psi}_{\gamma b}^{\gamma}\times])C_b^\gamma \tag{18}$$

Similarly,

$$C_b^\gamma = (\mathcal{I} + [\delta\vec{\psi}_{\gamma b}^{\gamma}\times])\hat{C}_b^\gamma \tag{19}$$

Similarly the measuremed specific force and angular rate may be written in terms of the estimates as

$$\tilde{\vec{f}}^{\,b}_{ib} = \hat{\vec{f}}^{\,b}_{ib} + \Delta\hat{\vec{f}}^{\,b}_{ib} \tag{20}$$

$$\tilde{\vec{\omega}}^{\,b}_{ib} = \hat{\vec{\omega}}^{\,b}_{ib} + \Delta\hat{\vec{\omega}}^{\,b}_{ib} \tag{21}$$

where $\hat{\vec{f}}^{\,b}_{ib}$ and $\hat{\vec{\omega}}^{\,b}_{ib}$ are the accelerometer and gyroscope estimated calibration values, respectively.

Since the sensor measurements are corrupted with errors, derive an error model describing the position, velocity, and attitude as a function of time.

$$\dot{C}_b^i = C_b^i \Omega_{ib}^b = \frac{d}{dt}\left[\left(\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times]\right)\hat{C}_b^i\right] =$$

$$\dot{C}_b^i = C_b^i \Omega_{ib}^b \;= \frac{d}{dt}\left[\left(\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times]\right)\hat{C}_b^i\right] =$$

$$\left(\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times]\right)\hat{C}_b^i\Omega_{ib}^b = [\delta\dot{\vec{\psi}}^{\,i}_{ib}\times]\hat{C}_b^i + \left(\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times]\right)\dot{\hat{C}}_b^i =$$

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$$(\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\hat{C}_b^i(\hat{\Omega}_{ib}^b + \delta\Omega_{ib}^b) =$$

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$$(\mathcal{I} + [\delta\vec{\psi}\,_{ib}^i\times])\hat{C}_b^i(\hat{\Omega}_{ib}^b + \delta\Omega_{ib}^b) =$$

$$= \hat{C}_b^i\delta\Omega_{ib}^b + (\mathcal{I} + [\delta\vec{\psi}\,_{ib}^i\times])\hat{C}_b^i\hat{\Omega}_{ib}^b$$

$$\because [\delta\vec{\psi}\,_{ib}^i\times]\delta\Omega_{ib}^b \approx 0$$

$$\dot{C}_b^i = C_b^i \Omega_{ib}^b = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\hat{C}_b^i\right] =$$

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$$[\delta\dot{\vec{\psi}}^{\,i}_{ib}\times] = \hat{C}_b^i\delta\Omega_{ib}^b\hat{C}_i^b = [\hat{C}_b^i\delta\vec{\omega}^{\,b}_{ib}\times] \tag{22}$$

$$\dot{C}_b^i = C_b^i \Omega_{ib}^b = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\hat{C}_b^i \right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\hat{C}_b^i \Omega_{ib}^b = [\delta\dot{\vec{\psi}}^{\,i}_{ib}\times]\hat{C}_b^i + (\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\dot{\hat{C}}_b^i =$$

$$(\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\hat{C}_b^i(\hat{\Omega}_{ib}^b + \delta\Omega_{ib}^b) =$$

$$= \hat{C}_b^i \delta\Omega_{ib}^b + (\mathcal{I} + [\delta\vec{\psi}^{\,i}_{ib}\times])\hat{C}_b^i \hat{\Omega}_{ib}^b$$

$$[\delta\dot{\vec{\psi}}^{\,i}_{ib}\times] = \hat{C}_b^i \delta\Omega_{ib}^b \hat{C}_i^b = [\hat{C}_b^i \delta\vec{\omega}^{\,b}_{ib}\times] \tag{22}$$

$$\delta\dot{\vec{\psi}}^{\,i}_{ib} = \hat{C}_b^i \delta\vec{\omega}^{\,b}_{ib} \tag{23}$$

$$\dot{\vec{v}}^{\,i}_{ib} = C^i_b \vec{f}^{\,b}_{ib} + \vec{\gamma}^{\,i}_{ib} \tag{24}$$

$$\dot{\vec{v}}{}^i_{ib} = C^i_b \vec{f}{}^b_{ib} + \vec{\gamma}{}^i_{ib} \tag{24}$$

$$\dot{\hat{\vec{v}}}{}^i_{ib} = \hat{C}^i_b \hat{\vec{f}}{}^b_{ib} + \hat{\vec{\gamma}}{}^i_{ib} \tag{25}$$

$$\hat{\vec{f}}_{ib}^{\,b} = \tilde{\vec{f}}_{ib}^{\,b} - \Delta\hat{\vec{f}}_{ib}^{\,b} = \vec{f}_{ib}^{\,b} + \Delta\vec{f}_{ib}^{\,b} - \Delta\hat{\vec{f}}_{ib}^{\,b} = \vec{f}_{ib}^{\,b} + \Delta_e\vec{f}_{ib}^{\,b} = \vec{f}_{ib}^{\,b} - \delta\vec{f}_{ib}^{\,b}$$

$$\dot{\vec{v}}_{ib}^{\,i} = C_b^i\vec{f}_{ib}^{\,b} + \vec{\gamma}_{ib}^{\,i} \tag{24}$$

$$\dot{\hat{\vec{v}}}_{ib}^{\,i} = \hat{C}_b^i\hat{\vec{f}}_{ib}^{\,b} + \hat{\vec{\gamma}}_{ib}^{i} \tag{25}$$

$$\hat{\vec{f}}^{\,b}_{ib} = \tilde{\vec{f}}^{\,b}_{ib} - \Delta\hat{\vec{f}}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} + \Delta\vec{f}^{\,b}_{ib} - \Delta\hat{\vec{f}}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} + \Delta_e\vec{f}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} - \delta\vec{f}^{\,b}_{ib}$$

$$\dot{\vec{v}}^{\,i}_{ib} = C^i_b\vec{f}^{\,b}_{ib} + \vec{\gamma}^{\,i}_{ib} \tag{24}$$

$$\dot{\hat{\vec{v}}}^{\,i}_{ib} = \hat{C}^i_b\hat{\vec{f}}^{\,b}_{ib} + \hat{\vec{\gamma}}^{i}_{ib} = (\mathcal{I} - [\delta\vec{\psi}^{\,i}_{ib}\times])C^i_b(\vec{f}^{\,b}_{ib} + \Delta_e\vec{f}^{\,b}_{ib}) + \hat{\vec{\gamma}}^{i}_{ib} \tag{25}$$

$$\hat{\vec{f}}^{\,b}_{ib} = \tilde{\vec{f}}^{\,b}_{ib} - \Delta\hat{\vec{f}}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} + \Delta\vec{f}^{\,b}_{ib} - \Delta\hat{\vec{f}}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} + \Delta_e\vec{f}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} - \delta\vec{f}^{\,b}_{ib}$$

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$$\begin{aligned}\delta\dot{\vec{v}}^{\,i}_{ib} = \dot{\vec{v}}^{\,i}_{ib} - \dot{\hat{\vec{v}}}^{i}_{ib} &= [\delta\vec{\psi}^{\,i}_{ib}\times]C^i_b\vec{f}^{\,b}_{ib} + \hat{C}^i_b\delta\vec{f}^{\,b}_{ib} + \delta\vec{\gamma}^{\,i}_{ib} \\ &= [\delta\vec{\psi}^{\,i}_{ib}\times]\hat{C}^i_b\hat{\vec{f}}^{\,b}_{ib} + \hat{C}^i_b\delta\vec{f}^{\,b}_{ib} + \delta\vec{\gamma}^{\,i}_{ib}\end{aligned}$$

$$\hat{\vec{f}}^{\,b}_{ib} = \tilde{\vec{f}}^{\,b}_{ib} - \Delta\hat{\vec{f}}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} + \Delta\vec{f}^{\,b}_{ib} - \Delta\hat{\vec{f}}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} + \Delta_e\vec{f}^{\,b}_{ib} = \vec{f}^{\,b}_{ib} - \delta\vec{f}^{\,b}_{ib}$$

$$\dot{\vec{v}}^{\,i}_{ib} = C^i_b\vec{f}^{\,b}_{ib} + \vec{\gamma}^{\,i}_{ib} \tag{24}$$

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$$\delta\dot{\vec{v}}^{\,i}_{ib} = \dot{\vec{v}}^{\,i}_{ib} - \dot{\hat{\vec{v}}}^{i}_{ib} = [\delta\vec{\psi}^{\,i}_{ib}\times]C^i_b\vec{f}^{\,b}_{ib} + \hat{C}^i_b\delta\vec{f}^{\,b}_{ib} + \delta\vec{\gamma}^{\,i}_{ib}$$
$$= [\delta\vec{\psi}^{\,i}_{ib}\times]\hat{C}^i_b\hat{\vec{f}}^{\,b}_{ib} + \hat{C}^i_b\delta\vec{f}^{\,b}_{ib} + \delta\vec{\gamma}^{\,i}_{ib}$$

$$\delta\dot{\vec{v}}^{\,i}_{ib} = -[\hat{C}^i_b\hat{\vec{f}}^{\,b}_{ib}\times]\delta\vec{\psi}^{\,i}_{ib} + \hat{C}^i_b\delta\vec{f}^{\,b}_{ib} + \delta\vec{\gamma}^{\,i}_{ib} \tag{26}$$

$$\vec{\gamma}^{\,i}_{ib} \approx \frac{(r^e_{eS}(L_b))^2}{(r^e_{eS}(L_b) + h_b)^2} + \vec{\gamma}^{\,i}_0(L_b) \tag{27}$$

Assuming $h_b \ll r^e_{eS}$

$$\delta\vec{\gamma}^{\,i}_{ib} \approx -2\frac{(h_b - \hat{h}_b)}{r^e_{eS}(\hat{L}_b)} g_0(\hat{L}_b)\hat{\vec{u}}^i_D \tag{28}$$

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Then converting from curvlinear coordinates to ECI

$$\delta\vec{\gamma}^{\,i}_{ib} \approx \frac{2g_0(\hat{L}_b)}{r^e_{eS}(\hat{L}_b)} \frac{\hat{\vec{r}}^{\,i}_{ib}}{|\hat{\vec{r}}^{\,i}_{ib}|^2} (\hat{\vec{r}}^{\,i}_{ib})^T \delta\vec{r}^{\,i}_{ib} \tag{29}$$

$$\dot{\vec{r}}^{\,i}_{ib} = \vec{v}^{\,i}_{ib} \tag{30}$$

$$\dot{\vec{r}}^{\,i}_{ib} = \vec{v}^{\,i}_{ib} \tag{30}$$

$$\delta\dot{\vec{r}}^{\,i}_{ib} = \delta\vec{v}^{\,i}_{ib} \tag{31}$$

$$
\begin{pmatrix} \delta\dot{\vec{\psi}}^{\,i}_{ib} \\ \delta\dot{\vec{v}}^{\,i}_{ib} \\ \delta\dot{\vec{r}}^{\,i}_{ib} \end{pmatrix} = \begin{bmatrix} 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ -[\hat{C}^i_b \hat{\vec{f}}^{\,b}_{ib}\times] & 0_{3\times3} & \frac{2g_0(\hat{L}_b)}{r^e_{eS}(\hat{L}_b)} \frac{\hat{\vec{r}}^{\,i}_{ib}}{|\hat{\vec{r}}^{\,i}_{ib}|^2} (\hat{\vec{r}}^{\,i}_{ib})^T \\ 0_{3\times3} & \mathcal{I}_{3\times3} & 0_{3\times3} \end{bmatrix} \begin{pmatrix} \delta\vec{\psi}^{\,i}_{ib} \\ \delta\vec{v}^{\,i}_{ib} \\ \delta\vec{r}^{\,i}_{ib} \end{pmatrix} + \begin{bmatrix} 0 & \hat{C}^i_b \\ \hat{C}^i_b & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \delta\vec{f}^{\,b}_{ib} \\ \delta\vec{\omega}^{\,b}_{ib} \end{pmatrix} \tag{32}
$$

Summary - in terms of $\Delta_e \vec{f}^{\,b}_{ib}, \Delta_e \vec{\omega}^{\,b}_{ib}$

$$\begin{pmatrix} \delta\dot{\vec{\psi}}^{\,i}_{ib} \\ \delta\dot{\vec{v}}^{\,i}_{ib} \\ \delta\dot{\vec{r}}^{\,i}_{ib} \end{pmatrix} = \begin{bmatrix} 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ -[\hat{C}^i_b \hat{\vec{f}}^{\,b}_{ib}\times] & 0_{3\times3} & \frac{2g_0(\hat{L}_b)}{r^e_{eS}(\hat{L}_b)} \frac{\hat{\vec{r}}^{\,i}_{ib}}{|\hat{\vec{r}}^{\,i}_{ib}|^2}(\hat{\vec{r}}^{\,i}_{ib})^T \\ 0_{3\times3} & \mathcal{I}_{3\times3} & 0_{3\times3} \end{bmatrix} \begin{pmatrix} \delta\vec{\psi}^{\,i}_{ib} \\ \delta\vec{v}^{\,i}_{ib} \\ \delta\vec{r}^{\,i}_{ib} \end{pmatrix} + \begin{bmatrix} 0 & -\hat{C}^i_b \\ -\hat{C}^i_b & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \Delta_e \vec{f}^{\,b}_{ib} \\ \Delta_e \vec{\omega}^{\,b}_{ib} \end{pmatrix} \tag{33}$$