EE 565: Position, Navigation and Timing
Navigation Mathematics: Kinematics (Earth Surface & Gravity Models)

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- The earth can be modeled as an oblate spheroid

  - A circular cross section when viewed from the polar axis (top view)

  - An elliptical cross–section when viewed perpendicular to the polar axis (side view)
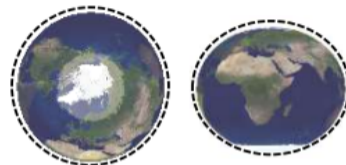
Ratio exaggerated

- This ellipsoid (i.e., oblate spheroid) is an approximation of the "geoid"

- The geoid is a gravitational equipotential surface which "best" fits (in the least square sense) the mean sea level

$$H = h - N$$

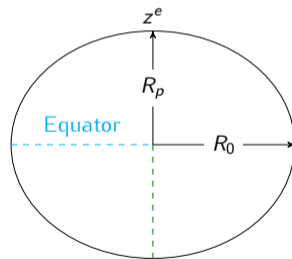Orthometric Height = Ellipsoidal Height from GPS - Geoid Height
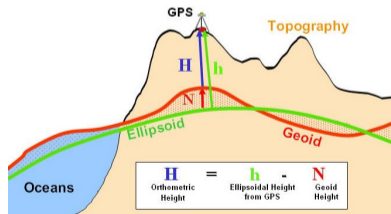
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- WGS 84 provides as model of the earth's geoid
  - More recently replace by EGM 2008
- The equatorial radius radius $R_0 = 6,378,137.0$m
- The polar radius radius $R_p = 6,356,752.3142$m
- Eccentricity of the ellipsoid

$$e = \sqrt{1 - \frac{R_p^2}{R_0^2}} \approx 0.0818$$

- Flattening of the ellipsoid

$$f = \frac{R_0 - R_p}{R_0} \approx \frac{1}{298}$$

- We can define a position "near" the earth's surface in terms of latitude, longitude, and height
  - Geocentric latitude intersects the center of mass of the earth
  - Geodetic latitude (L) is the angle between the normal to the ellipsoid and the equatorial plane

z^e

Reference Ellipsoid

Geocentric Latitude

Geodetic Latitude
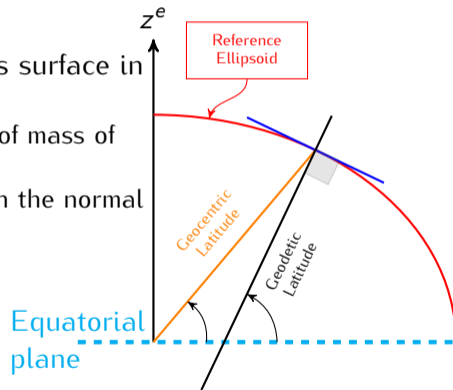
Equatorial plane

- The longitude (λ) is the angle from the x-axis of the ECEF frame to the projection of $\vec{r}_{eb}$ onto the equatorial plane
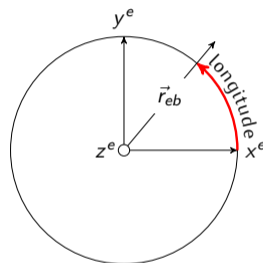
- The geocentric radius is the distance from center of the Earth to the point S
- The geodetic (or ellipsoidal) height (h) is the distance along the normal from the ellipsoid to the body
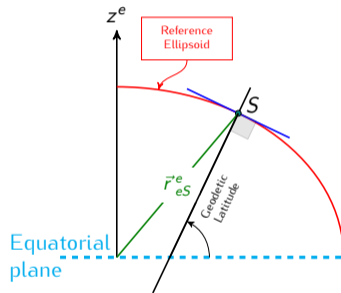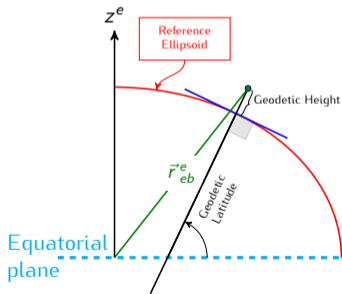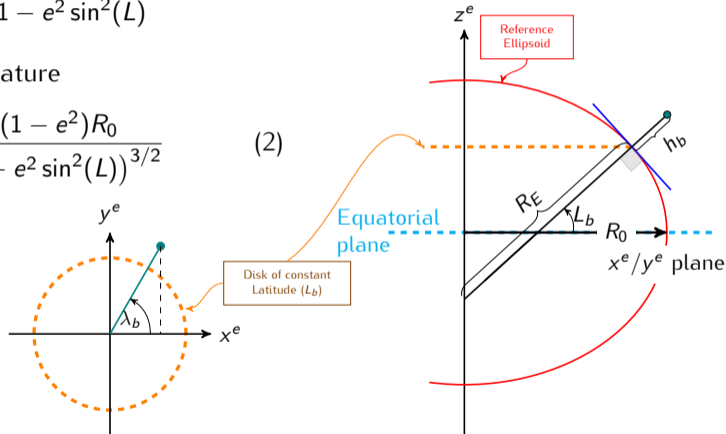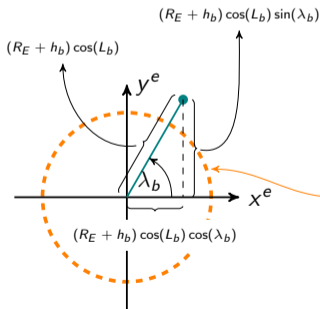
- Transverse radius of curvature

$$R_E(L) = \frac{R_0}{\sqrt{1 - e^2 \sin^2(L)}} \qquad (1)$$

- Meridian radius of curvature

$$R_N(L) = \frac{(1 - e^2)R_0}{\left(1 - e^2 \sin^2(L)\right)^{3/2}} \qquad (2)$$

$$\vec{r}^{\,e}_{eb} = \begin{bmatrix} x^e_{eb} \\ y^e_{eb} \\ z^e_{eb} \end{bmatrix} = \begin{bmatrix} (R_E + h_b)\cos(L_b)\cos(\lambda_b) \\ (R_E + h_b)\cos(L_b)\sin(\lambda_b) \\ (R_E(1 - e^2) + h_b)\sin(L_b) \end{bmatrix}$$

$(R_E + h_b)\cos(L_b)\sin(\lambda_b)$

$(R_E + h_b)\cos(L_b)$

y^e

λ_b

x^e

$(R_E + h_b)\cos(L_b)\cos(\lambda_b)$

Disk of constant Latitude (L_b)

z^e

$(R_E + h_b)\cos(L_b)$

$(R_E(1 - e^2) + h_b)\sin(L_b)$

h_b

R_E

L_b

R_0

x^e/y^e plane
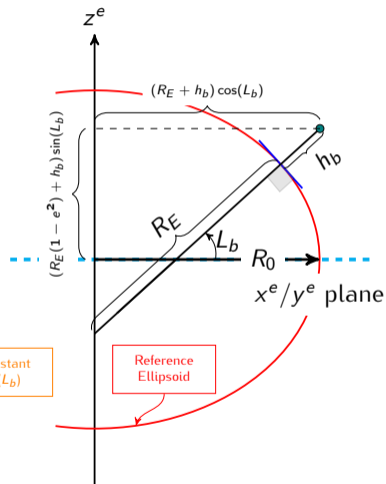
Reference Ellipsoid

- Specific force ($\vec{f}_{ib}$)
  - Non–gravitational force per unit mass (unit of acceleration)
    - Accelerometers measure specific force
- Specific force sensed when stationary (*wrt* earth) is referred to as the acceleration due to gravity ($\vec{g}_b$)
  - Actually, the reaction to this force
- Gravitational force (γ_{ib}) is result of mass attraction
  - The gravitational mass attraction force is different from the acceleration due to gravity

- Relationship between specific force, inertial acceleration, and gravitational attraction

Specific force

$$\vec{f}_{ib} = \vec{a}_{ib} - \vec{\gamma}_{ib} \tag{3}$$

- When stationary on the surface of the earth
  - A fixed point in a rotating frame

$$\ddot{\vec{r}}_{02}^{\,0}(t) = \dot{\vec{\omega}}_{01}^{\,0} \times \vec{r}_{12}^{\,0}(t) \qquad + \vec{\omega}_{01}^{\,0} \times \left(\vec{\omega}_{01}^{\,0} \times \vec{r}_{12}^{\,0}(t) \right)$$

$$\dot{\omega} = 0$$

  - Consider frame {0} to be the {i} frame, {1}={e}, and {2}={b} gives

$$\ddot{\vec{r}}_{ib}^{\,i}(t) = \vec{\omega}_{ie}^{\,i} \times \left(\vec{\omega}_{ie}^{\,i} \times \vec{r}_{eb}^{\,i}(t) \right)$$

  - coordinatizing in the e-frame

$$\ddot{\vec{r}}_{ib}^{\,e}(t) = \vec{\omega}_{ie}^{\,e} \times \left(\vec{\omega}_{ie}^{\,e} \times \vec{r}_{eb}^{\,e}(t) \right)$$

- Thus, when stationary on the surface of the earth the acceleration is due to centrifugal force

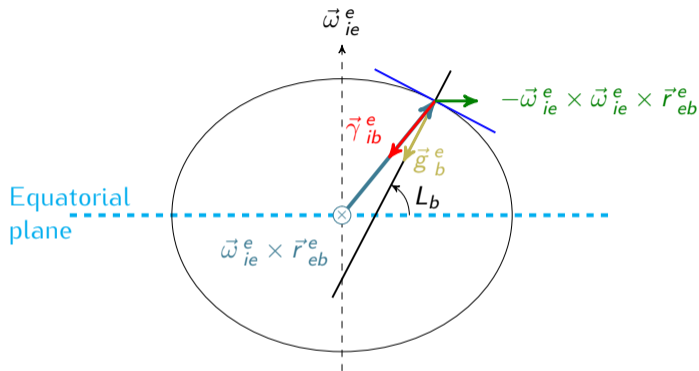$$\vec{a}^{\,e}_{ib} = \Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb}$$

- Therefore, the acceleration due to gravity is

- Thus, when stationary on the surface of the earth the acceleration is due to centrifugal force

$$\vec{a}_{ib}^{\,e} = \Omega_{ie}^{e}\Omega_{ie}^{e}\vec{r}_{eb}^{\,e}$$

- Therefore, the acceleration due to gravity is

$$\vec{g}_{b}^{\,e} = -\left.\vec{f}_{ib}^{\,}\right|_{\vec{v}_{eb}^{\,e}=0} = -\Omega_{ie}^{e}\Omega_{ie}^{e}\vec{r}_{eb}^{\,e} + \vec{\gamma}_{ib}^{\,e} \tag{4}$$

$$\vec{g}_b^e = -\Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^e + \vec{\gamma}_{ib}^e$$

1g   ≈3mg   ≈1g

- Now, $\vec{\omega}_{ie}^{\,e} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega_{ie}$ and hence, $\Omega_{ie}^{e} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \omega_{ie}$, and thus

$$\vec{g}_{b}^{\,e} = \vec{\gamma}_{ib}^{\,e} + \omega_{ie}^{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{r}_{eb}^{\,e}$$

- The WGS 84 model of acceleration due to gravity (on the ellipsoid) can be approximated by (Somigliana model)

$$g_0(L_b) = 9.7803253359 \frac{(1 + 0.001931853 \sin^2(L))}{\sqrt{1 - e^2 \sin^2(L)}} \tag{5}$$

$$g_{b,D}^n = g_0(L_b, h_b) \left\{ 1 - \frac{2}{R_0} \left[1 + f(1 - 2\sin^2 L_b) + \frac{\omega_{ie}^2 R_0^2 R_p}{\mu} \right] h_b + \frac{3}{R_0^2} h_b^2 \right\} \tag{6}$$

where $\mu = 3.986004418 \times 10^{14}$ m^3/s^2 is the WGS 84 Earth's gravitational constant.

- On March 17, 2002 NASA launched the Gravity Recovery and Climate Experiment (GRACE) which led to the development of some of the most precise Earth gravity models.

Earth's Gravity Field Anomalies (milligals)

-50 -40 -30 -20 -10 0 10 20 30 40 50