Lecture
Kalman Filtering Example
EE 565: Position, Navigation, and Timing

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.1

1 Kalman Filter

Review: System Model

$$\dot{\vec{x}}(t) = F(t)\vec{x}(t) + G(t)\vec{w}(t) \tag{1}$$

$$\vec{y}(t) = H(t)\vec{x}(t) + \vec{v}(t) \tag{2}$$

System Discretization

$$\Phi_{k-1} = e^{F_{k-1}\tau_s} \approx \mathcal{I} + F_{k-1}\tau_s \tag{3}$$

where F_{k-1} is the average of F at times t and $t - \tau_s$, and first order approximation is used. Leading to

$$\vec{x}_k = \Phi_{k-1}\ \vec{x}_{k-1} + \vec{w}_{k-1} \tag{4}$$

$$\vec{z}_k = H_k\ \vec{x}_k + \vec{v}_k \tag{5}$$

where Φ_{k-1} is $(n \times n)$ transition matrix relating $\vec{x}_{k-1}$ to $\vec{x}_k$, H_k is $(m \times n)$ matrix provides noiseless connection between measurement and state vectors.

.2

Review: Assumptions

- $\vec{w}_k$ and $\vec{v}_k$ are drawn from a Gaussian distribution, uncorrelated have zero mean and statistically independent.

$$\mathbb{E}\{\vec{w}_k\vec{w}_i^T\} = \begin{cases} Q_k & i = k \\ 0 & i \neq k \end{cases} \tag{6}$$

$$\mathbb{E}\{\vec{v}_k\vec{v}_i^T\} = \begin{cases} R_k & i = k \\ 0 & i \neq k \end{cases} \tag{7}$$

$$\mathbb{E}\{\vec{w}_k\vec{v}_i^T\} = \begin{cases} 0 & \forall i, k \end{cases} \tag{8}$$

- State covariance matrix

$$Q_{k-1} \approx \frac{1}{2}\left[\Phi_{k-1}G_{k-1}Q(t_{k-1})G_{k-1}^T\Phi_{k-1}^T + G_{k-1}Q(t_{k-1})G_{k-1}^T\right]\tau_s \tag{9}$$

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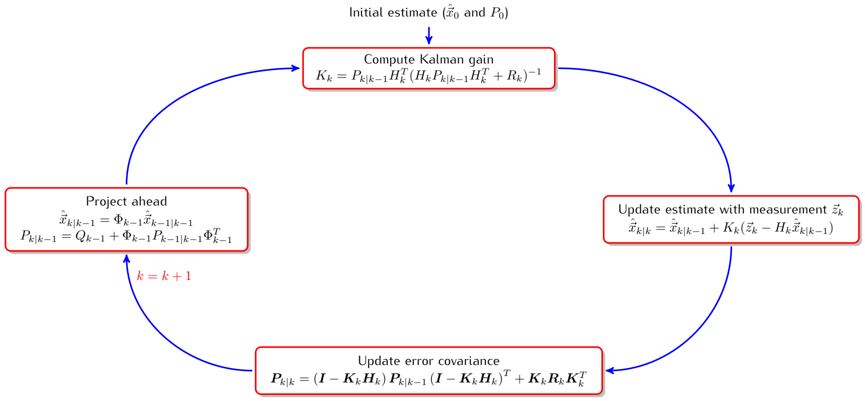
Review: Kalman filter data flow



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Remarks

- Kalman filter (KF) is optimal under the assumptions that the system is linear and the noise is uncorrelated
- Under these assumptions KF provides an unbiased and minimum variance estimate.
- If the Gaussian assumptions is not true, Kalman filter is biased and not minimum variance.
- If the noise is correlated we can augment the states of the system to maintain the uncorrelated requirement of the system noise.

.5

2 State Augmentation

Correlated State Noise

Given a state space system

$$\dot{\vec{x}}_1(t) = F_1(t)\vec{x}_1(t) + G_1(t)\vec{w}_1(t)$$

$$\vec{y}_1(t) = H_1(t)\vec{x}_1(t) + \vec{v}_1(t)$$

As we have seen the noise $\vec{w}_1(t)$ may be non-white, e.g., correlated Gaussian noise, and as such may be modeled as

$$\dot{\vec{x}}_2(t) = F_2(t)\vec{x}_2(t) + G_2(t)\vec{w}_2(t)$$

$$\vec{w}_1(t) = H_2(t)\vec{x}_2(t)$$

.6

Correlated State Noise

Define a new augmented state

$$\vec{x}_{aug} = \begin{pmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{pmatrix} \tag{10}$$

therefore,

$$\dot{\vec{x}}_{aug} = \begin{pmatrix} \dot{\vec{x}}_1(t) \\ \dot{\vec{x}}_2(t) \end{pmatrix} = \begin{pmatrix} F_1(t) & G_1H_2(t) \\ 0 & F_2(t) \end{pmatrix} \begin{pmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ G_2(t) \end{pmatrix} \vec{w}_2(t) \tag{11}$$

and

$$\vec{y}(t) = \begin{pmatrix} H_1(t) & 0 \end{pmatrix} \begin{pmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{pmatrix} + \vec{v}_1(t) \tag{12}$$

.7

Correlated Measurement Noise

Given a state space system

$$\dot{\vec{x}}_1(t) = F_1(t)\vec{x}_1(t) + G_1(t)\vec{w}(t)$$

$$\vec{y}_1(t) = H_1(t)\vec{x}_1(t) + \vec{v}_1(t)$$

In this case the measurement noise $\vec{v}_1$ may be correlated

$$\dot{\vec{x}}_2(t) = F_2(t)\vec{x}_2(t) + G_2(t)\vec{v}_2(t)$$

$$\vec{v}_1(t) = H_2(t)\vec{x}_2(t)$$

.8

Correlated Measurement Noise

Define a new augmented state

$$\vec{x}_{aug} = \begin{pmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{pmatrix} \tag{13}$$

therefore,

$$\dot{\vec{x}}_{aug} = \begin{pmatrix} \dot{\vec{x}}_1(t) \\ \dot{\vec{x}}_2(t) \end{pmatrix} = \begin{pmatrix} F_1(t) & 0 \\ 0 & F_2(t) \end{pmatrix} \begin{pmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{pmatrix} + \begin{pmatrix} G_1(t) & 0 \\ 0 & G_2(t) \end{pmatrix} \begin{pmatrix} \vec{w}(t) \\ \vec{v}_2(t) \end{pmatrix} \tag{14}$$

and

$$\vec{y}(t) = \begin{pmatrix} H_1(1) & H_2(t) \end{pmatrix} \begin{pmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{pmatrix} \tag{15}$$

.9

3 Example

Design Example

You are to design a system that estimates the position and velocity of a moving point in a straight line. You have:

1. an accelerometer corrupted with noise
2. an aiding sensor allowing you to measure absolute position that is also corrupted with noise.

.10

Specification

- Sampling Rate Fs = 100Hz.
- Accelerometer specs
    1. VRW = $1mg/\sqrt{Hz}$.
    2. BI = $7mg$ with correlation time 6s.
- Position measurement is corrupted with WGN. $\sim \mathcal{N}(0, \sigma_p^2)$, where $\sigma_p = 2.5$m

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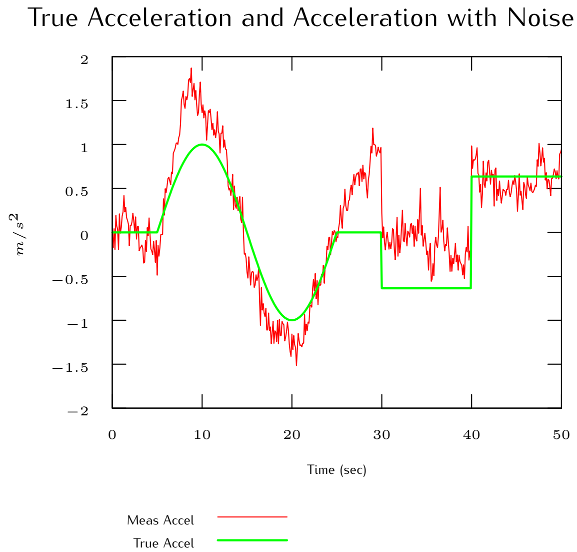
Input - Acceleration



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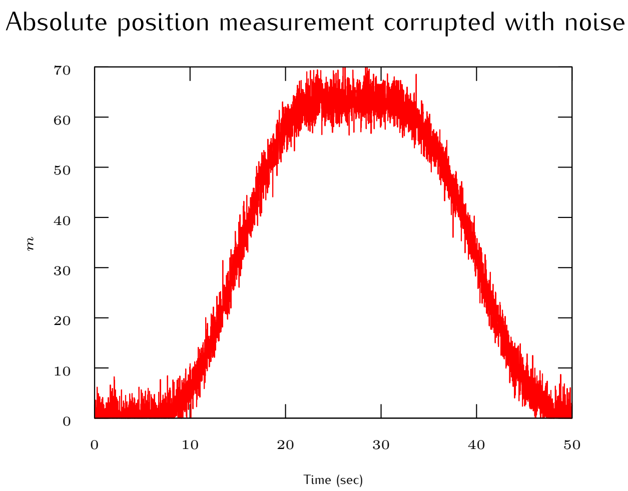
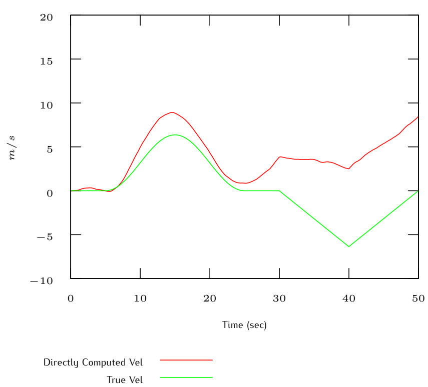
Aiding Position Measurement



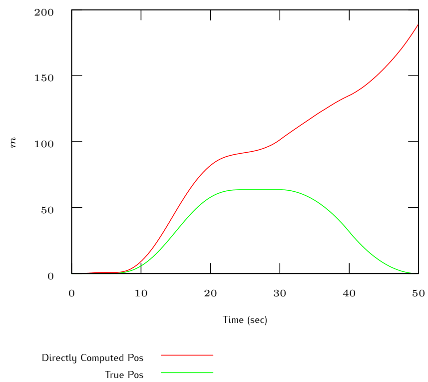
.13

Computed Position and Velocity

Using only the acceleration measurement and an integration approach to compute the velocity, then integrate again to get position.



Velocity



Position

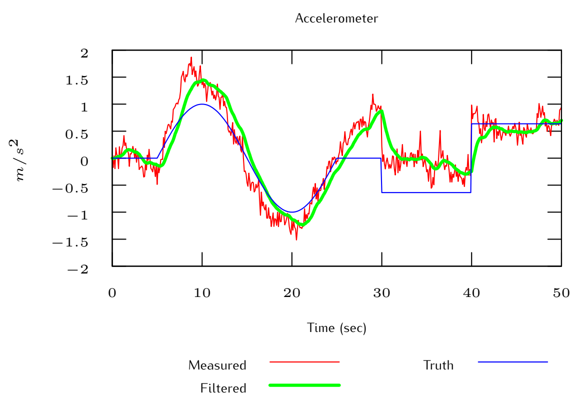
.14

Different Approaches

1. Clean up the noisy input to the system by filtering
2. Use Kalman filtering techniques with
    - A model of the system dynamics (too restrictive)
    - A model of the error dynamics and correct the system output in
        - open-loop configuration, or
        - closed-loop configuration.

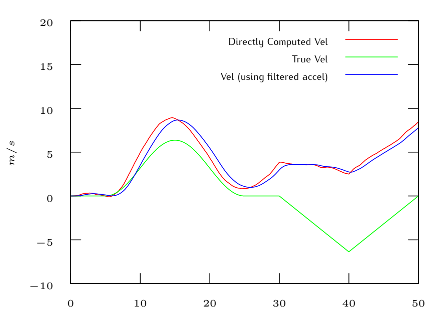
.15

Approach 1 — Filtered input   Filtered Accel Measurement

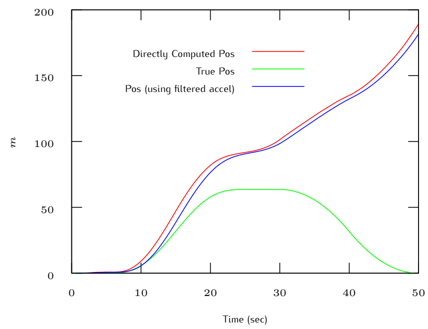


.16

Approach 1 — Filtered input   Position and Velocity



Velocity



Position

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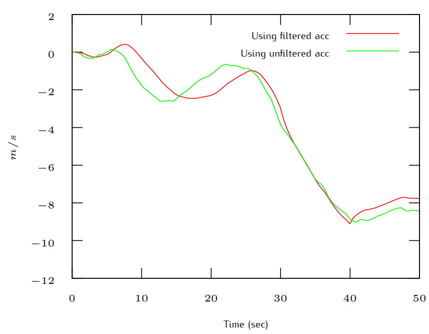
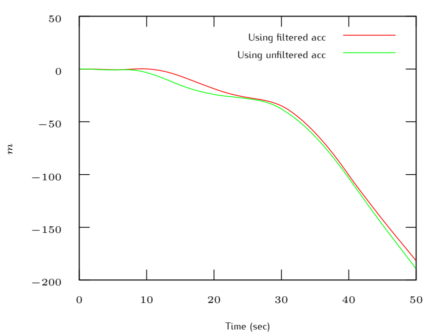
Approach 1 — Filtered input Position and Velocity Errors



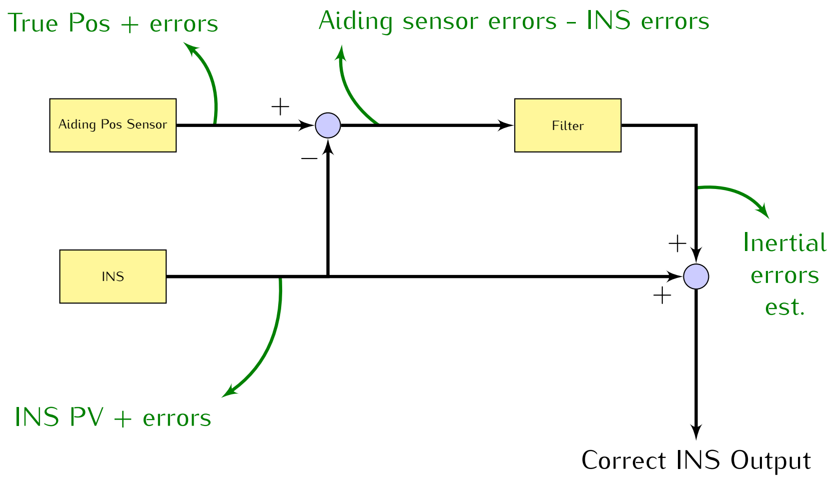
Velocity Error



Position Error

.18

Open-Loop Integration



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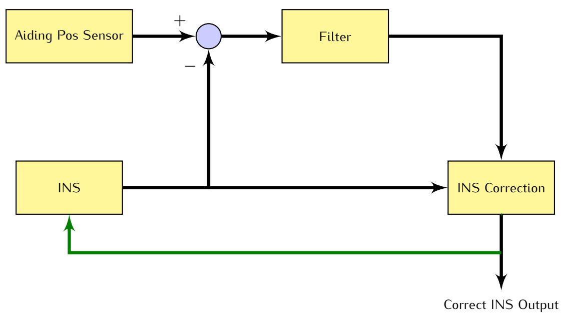
Closed-Loop Integration

If error estimates are fedback to correct the INS mechanization, a reset of the state estimates becomes necessary.



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Covariance Matrices

- State noise covariance matrix (continuous)

$$\mathbb{E}\{\vec{w}(t)\vec{w}^T(\tau)\} = Q(t)\delta(t-\tau)$$

- State noise covariance matrix (discrete)

$$\mathbb{E}\{\vec{w}_k\vec{w}_i^T\} = \begin{cases} Q_k & i = k \\ 0 & i \neq k \end{cases}$$

- Measurement noise covariance matrix

$$\mathbb{E}\{\vec{v}_k\vec{v}_i^T\} = \begin{cases} R_k & i = k \\ 0 & i \neq k \end{cases}$$

- Initial error covariance matrix

$$P_0 = \mathbb{E}\{(\vec{x}_0 - \hat{\vec{x}}_0)(\vec{x}_0 - \hat{\vec{x}}_0)^T\} = \mathbb{E}\{\vec{e}_0\hat{\vec{e}}_0^T\}$$

.21

System Modeling

The position, velocity and acceleration may be modeled using the following kinematic model.

$$\begin{aligned} \dot{p}(t) &= v(t) \\ \dot{v}(t) &= a(t) \end{aligned} \tag{16}$$

where $a(t)$ is the input. Therefore, our estimate of the position is $\hat{p}(t)$ that is the double integration of the acceleration. .22

Sensor Model

Assuming that the accelerometer sensor measurement may be modeled as

$$\tilde{a}(t) = a(t) + b(t) + w_a(t) \tag{17}$$

and the bias is Markov, therefore

$$\dot{b}(t) = -\frac{1}{T_c} b(t) + w_b(t) \tag{18}$$

where $w_a(t)$ and $w_b(t)$ are zero mean WGN with variances, respectively, $Fs \cdot VRW^2$

$$\mathbb{E}\{w_b(t) w_b(t+\tau)\} = Q_b(t)\delta(t-\tau) \tag{19}$$

$$Q_b(t) = \frac{2\sigma_{BI}^2}{T_c} \tag{20}$$

and T_c is the correlation time and σ_{BI} is the bias instability.

Make sure that the VRW and σ_{BI} are converted to have SI units. .23

Error Mechanization

Define error terms as

$$\delta p(t) = p(t) - \hat{p}(t), \tag{21}$$

$$\begin{aligned} \delta \dot{p}(t) &= \dot{p}(t) - \dot{\hat{p}}(t) \\ &= v(t) - \hat{v}(t) \\ &= \delta v(t) \end{aligned} \tag{22}$$

and

$$\begin{aligned} \delta \dot{v}(t) &= \dot{v}(t) - \dot{\hat{v}}(t) \\ &= a(t) - \hat{a}(t) \\ &= -b(t) - w_a(t) \end{aligned} \tag{23}$$

where $b(t)$ is modeled as shown in Eq. 18 .24

State Space Formulation

$$\begin{aligned} \dot{\vec{x}}(t) = \begin{pmatrix} \delta \dot{p}(t) \\ \delta \dot{v}(t) \\ \dot{b}(t) \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -\frac{1}{T_c} \end{pmatrix} \begin{pmatrix} \delta p(t) \\ \delta v(t) \\ b(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ w_a(t) \\ w_b(t) \end{pmatrix} \\ &= F(t)\vec{x}(t) + G(t)\vec{w}(t) \end{aligned} \tag{24}$$

.25

Covariance Matrix

- The continuous state noise covariance matrix $Q(t)$ is

$$Q(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & VRW^2 & 0 \\ 0 & 0 & \frac{2\sigma_{BI}^2}{Tc} \end{pmatrix} \tag{25}$$

- The measurement noise covariance matrix is $R = \sigma_p^2$, where σ_p is the standard deviation of the noise of the absolute position sensor.

.26

Discretization

Now we are ready to start the implementation but first we have to discretize the system.

$$\vec{x}(k+1) = \Phi(k)\vec{x}(k) + \vec{w}_d(k) \tag{26}$$

where

$$\Phi(k) \approx \mathcal{I} + Fdt \tag{27}$$

with the measurement equation

$$y(k) = H\vec{x} + w_p(k) = \delta p(k) + w_p(k) \tag{28}$$

where $H = [1\ 0\ 0]$. The discrete Q_d is approximated as

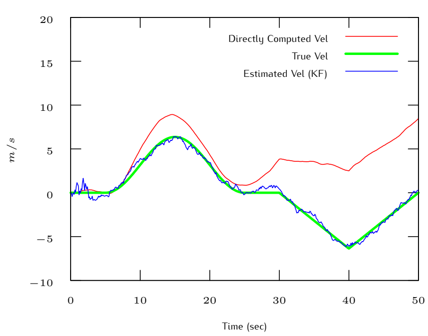
$$Q_{k-1} \approx \frac{1}{2}[\Phi_{k-1}G(t_{k-1})Q(t_{k-1})G^T(t_{k-1}))\Phi_{k-1}^T + G(t_{k-1})Q(t_{k-1})G^T(t_{k-1})]dt \tag{29}$$

.27

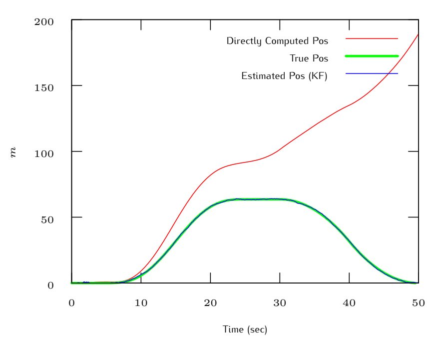
Approach 2 — Open-Loop Compensation Position and Velocity

Open-loop Correction

Best estimate = INS out (pos & vel) + KF est error (pos & vel)



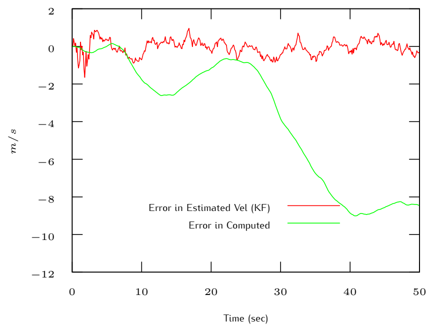
Velocity



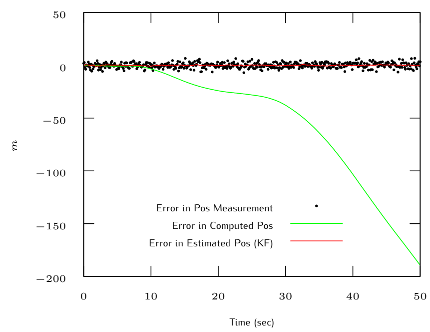
Position

.28

Approach 2 — Open-Loop Compensation Position and Velocity Errors



Velocity Error



Position Error

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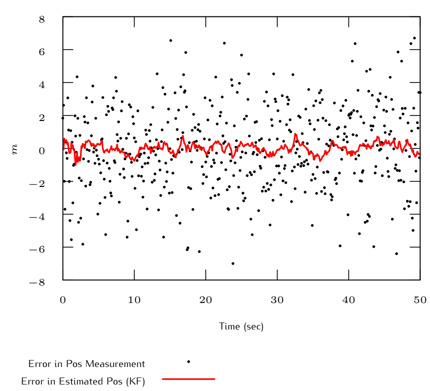
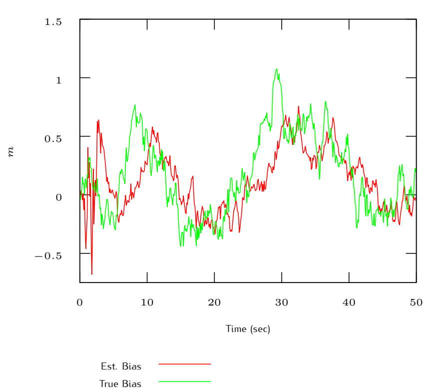
Approach 2 — Open-Loop Compensation Pos Error & Bias Estimate



Position Error



Bias

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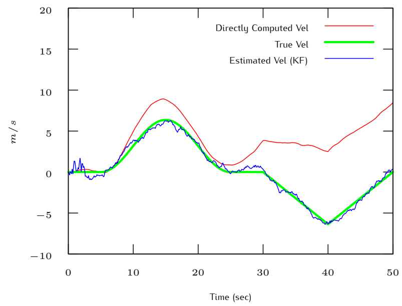
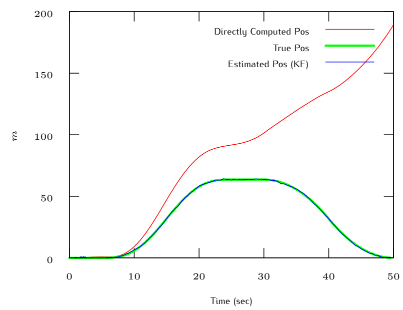
Approach 3 — Closed-Loop Compensation

Closed-loop Correction
Best estimate = INS out (pos,vel, & bias) + KF est error (pos, vel & bias) Use best estimate on next iteration of INS Accel estimate = accel meas - est bias Reset state estimates before next call to KF



Velocity



Position

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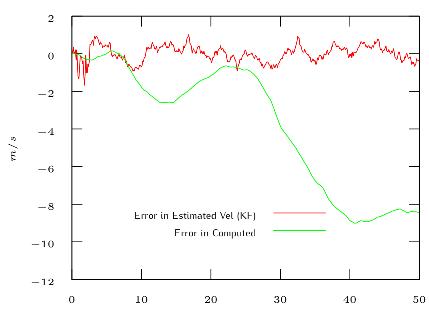
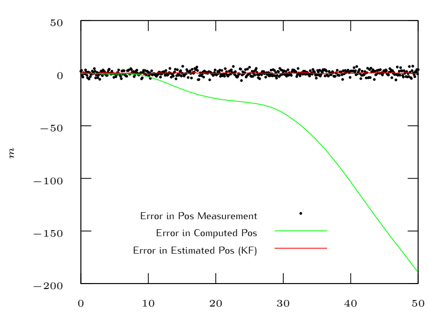
Approach 3 — Closed-Loop Compensation Position and Velocity Errors



Velocity Error



Position Error

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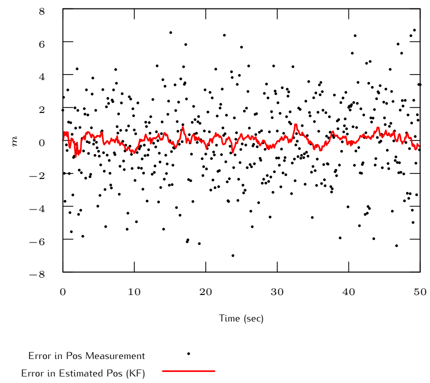
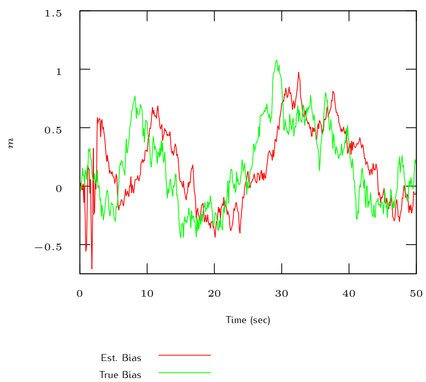
Approach 3 — Closed-Loop Compensation Pos Error & Bias Estimate



Position Error



Bias

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