Lecture

Navigation Equations: ECEF Mechanization

EE 565: Position, Navigation and Timing

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ECEF Mechanization

- Determine the position, velocity and attitude of the **body** frame *wrt* the **Earth** frame.
  - **Position** — Vector from the origin of the earth frame to the origin of the body frame resolved in the earth frame: $\vec{r}^{\,e}_{eb}$
  - **Velocity** — Velocity of the body frame *wrt* the earth frame resolved in the earth frame: $\vec{v}^{\,e}_{eb}$
  - **Attitude** — Orientation of the body frame *wrt* the earth frame: C^e_b

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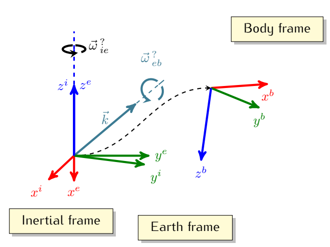
Attitude — Method A

- Body orientation frame at time "k" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$
- Start with angular velocity

$$\begin{aligned}\vec{\omega}^{\,e}_{ib} &= \vec{\omega}^{\,e}_{ie} + \vec{\omega}^{\,e}_{eb}\\ \vec{\omega}^{\,e}_{eb} &= C^e_b\vec{\omega}^{\,b}_{ib} - \vec{\omega}^{\,e}_{ie}\\ \Omega^e_{eb} &= C^e_b\Omega^b_{ib}C^b_e - \Omega^e_{ie}\end{aligned}$$

$$C^e_b(+) - C^e_b(-) \approx \Delta t\Omega^e_{eb}C^e_b(-)$$

$$\begin{aligned}C^e_b(+) &\approx C^e_b(-) + \Delta t\left(C^e_b\Omega^b_{ib}C^b_e - \Omega^e_{ie}\right)C^e_b(-)\\ &= C^e_b(-)\left(\mathcal{I} + \Omega^b_{ib}\Delta t\right) - \Omega^e_{ie}C^e_b(-)\Delta t\end{aligned}$$



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Attitude — Method B

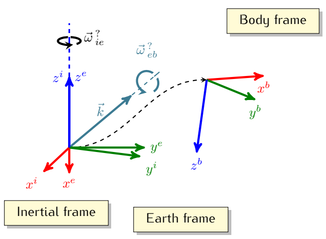
- Body orientation frame at time "k" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$
- Start with the angular velocity

$$\Omega^e_{eb} = C^e_b \Omega^b_{ib} C^b_e - \Omega^e_{ie}$$

$$C^e_b(+) = C^e_b(-) e^{\Omega^b_{eb}\Delta t} = e^{\Omega^e_{eb}\Delta t} C^e_b(-)$$

$$C^e_b(+) = \left[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1-\cos(\Delta\theta)]\,\mathfrak{K}^2\right] C^e_b(-)$$

$$e^{\Omega^e_{eb}\Delta t} = e^{\mathfrak{K}\theta}$$



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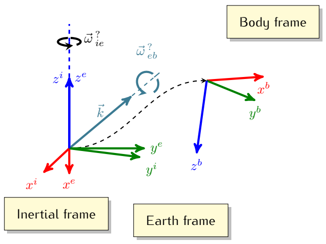
Attitude — Method C

- Body orientation frame at time "k" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$

$$\vec{\omega}^{\,e}_{eb}\Delta t = \vec{k}\Delta\theta$$

$$\bar{q}^{\,e}_{\,b}(+) = \Delta\bar{q}^{\,e}_{\,b} \otimes \bar{q}^{\,e}_{\,b}(-)$$

$$\Delta\bar{q}^{\,e}_{\,b} = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix}$$



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Need to periodically renormalize $\bar{q}$

Attitude Update— Summary

- High fidelity

$$C^e_b(+) = \left[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1-\cos(\Delta\theta)]\,\mathfrak{K}^2\right] C^e_b(-) \tag{1}$$

or

$$\bar{q}^{\,e}_{\,b}(+) = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix} \otimes \bar{q}^{\,e}_{\,b}(-) \tag{2}$$

- Low fidelity

$$C^e_b(+) \approx C^e_b(-)\left(\mathcal{I} + \Omega^b_{ib}\Delta t\right) - \Omega^e_{ie} C^e_b(-)\Delta t \tag{3}$$

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Steps 2–4

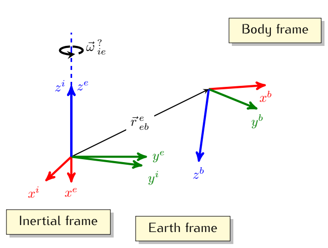
2. Specific force transformation
   - Simply coordinatize the specific force

$$\vec{f}^{\,e}_{\,ib} = C^e_b(+)\vec{f}^{\,b}_{\,ib} \tag{4}$$

3. Velocity update

$$\vec{r}^{\,i}_{\,ib} = \underbrace{\vec{r}^{\,i}_{\,ie}}_{0} + C^i_e \vec{r}^{\,e}_{\,eb} \Rightarrow \vec{r}^{\,e}_{\,eb} = C^e_i \vec{r}^{\,i}_{\,ib}$$

$$\begin{aligned} \vec{v}^{\,e}_{\,eb} &= \dot{\vec{r}}^{\,e}_{\,eb} \\ &= \dot{C}^e_i \vec{r}^{\,i}_{\,ib} + C^e_i \dot{\vec{r}}^{\,i}_{\,ib} \\ &= \Omega^e_{ei} C^e_i \vec{r}^{\,i}_{\,ib} + C^e_i \vec{v}^{\,i}_{\,ib} \\ &= -\Omega^e_{ie} \vec{r}^{\,e}_{\,eb} + C^e_i \vec{v}^{\,i}_{\,ib} \end{aligned}$$



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Steps 2–4

$$C_i^e \vec{v}_{ib}^{\,i} = \Omega_{ie}^e \vec{r}_{eb}^{\,e} + \vec{v}_{eb}^{\,e}$$

$$\vec{f}_{ib}^{\,?} = \vec{a}_{ib}^{\,?} - \vec{\gamma}_{ib}^{\,?}$$

$$\vec{a}_{ib}^{\,e} = \vec{f}_{ib}^{\,e} + \vec{\gamma}_{ib}^{\,e}$$

$$\vec{g}_b^{\,e} = \vec{\gamma}_{ib}^{\,e} - \Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^{\,e}$$

$$\vec{a}_{ib}^{\,e} = \vec{f}_{ib}^{\,e} + \vec{g}_b^{\,e} + \Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^{\,e}$$

$$\begin{aligned}
\vec{a}_{eb}^{\,e} = \dot{\vec{v}}_{eb}^{\,e} &= \frac{d}{dt}\left(-\Omega_{ie}^e \vec{r}_{eb}^{\,e} + C_i^e \vec{v}_{ib}^{\,i}\right) \\
&= -\Omega_{ie}^e \dot{\vec{r}}_{eb}^{\,e} + \dot{C}_i^e \vec{v}_{ib}^{\,i} + C_i^e \dot{\vec{v}}_{ib}^{\,i} \\
&= -\Omega_{ie}^e \vec{v}_{eb}^{\,e} + \Omega_{ei}^e C_i^e \vec{v}_{ib}^{\,i} + C_i^e \vec{a}_{ib}^{\,i} \\
&= -\Omega_{ie}^e \vec{v}_{eb}^{\,e} - \Omega_{ie}^e \left[\vec{v}_{eb}^{\,e} + \Omega_{ie}^e \vec{r}_{eb}^{\,e}\right] + C_i^e \vec{a}_{ib}^{\,i} \\
&= -2\Omega_{ie}^e \vec{v}_{eb}^{\,e} - \Omega_{ie}^e \Omega_{ie}^e \vec{r}_{eb}^{\,e} + \vec{a}_{ib}^{\,e} \\
&= -2\Omega_{ie}^e \vec{v}_{eb}^{\,e} + \vec{f}_{ib}^{\,e} + \vec{g}_b^{\,e}
\end{aligned}$$

$$\begin{aligned}
\vec{v}_{eb}^{\,e}(+) &= \vec{v}_{eb}^{\,e}(-) + \vec{a}_{eb}^{\,e}\Delta t \\
&= \vec{v}_{eb}^{\,e}(-) + \left[\vec{f}_{ib}^{\,e} + \vec{g}_b^{\,e} - 2\Omega_{ie}^i \vec{v}_{eb}^{\,e}(-)\right]\Delta t
\end{aligned} \quad (5)$$

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Steps 2–4

4. Position update
   - by simple numerical integration

$$\vec{r}_{eb}^{\,e}(+) = \vec{r}_{eb}^{\,e}(-) + \vec{v}_{eb}^{\,e}(-)\Delta t + \vec{a}_{eb}^{\,e}\frac{\Delta t^2}{2} \quad (6)$$

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ECEF Mechanization Summary

$$C_b^e(+) = \left[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1-\cos(\Delta\theta)]\,\mathfrak{K}^2\right] C_b^e(-)$$

or

$$C_b^e(+) \approx C_b^e(-)\left(\mathcal{I} + \Omega_{ib}^b \Delta t\right) - \Omega_{ie}^i C_b^e(-)\Delta t$$

or

$$\bar{q}_b^{\,e}(+) = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix} \otimes \bar{q}_b^{\,e}(-)$$

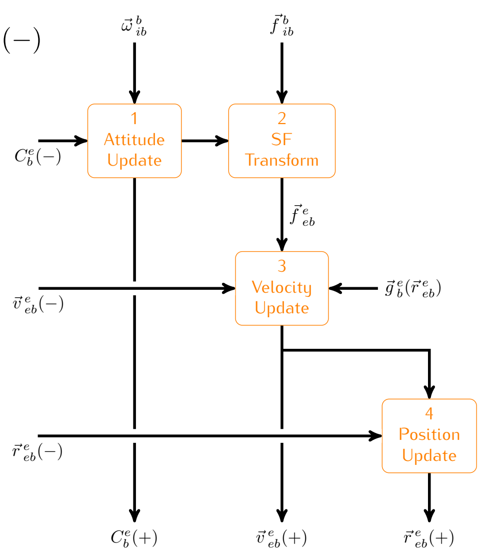
and

$$\vec{f}_{ib}^{\,e} = C_b^e(+)\vec{f}_{ib}^{\,b}$$

$$\vec{a}_{eb}^{\,e} = \vec{f}_{ib}^{\,e} + \vec{g}_b^{\,e} - 2\Omega_{ie}^i \vec{v}_{eb}^{\,e}(-)$$

$$\vec{v}_{eb}^{\,e}(+) = \vec{v}_{eb}^{\,e}(-) + \vec{a}_{eb}^{\,e}\Delta t$$

$$\vec{r}_{eb}^{\,e}(+) = \vec{r}_{eb}^{\,e}(-) + \vec{v}_{eb}^{\,e}(-)\Delta t + \vec{a}_{eb}^{\,e}\frac{\Delta t^2}{2}$$



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ECEF Mechanization — Continuous Case

- In continuous time notation
  - Attitude: $\dot{C}_b^e = C_b^e \Omega_{eb}^b$ or $\dot{\bar{q}}_b^e = \frac{1}{2}[\breve{\omega}_{eb}^{\,b}\circledast]\bar{q}_b^{\,e}(t)$
  - Velocity: $\dot{\vec{v}}_{eb}^{\,e} = C_b^e \vec{f}_{ib}^{\,b} + \vec{g}_b^{\,e} - 2\Omega_{ie}^i \vec{v}_{eb}^{\,e}$
  - Position: $\dot{\vec{r}}_{ib}^{\,e} = \vec{v}_{eb}^{\,e}$

- In State-space notation

$$\begin{bmatrix} \dot{\vec{r}}^{\,e}_{eb} \\ \dot{\vec{v}}^{\,e}_{eb} \\ \dot{C}^i_b \end{bmatrix} = \begin{bmatrix} \vec{v}^{\,e}_{eb} \\ C^e_b \vec{f}^{\,b}_{ib} + \vec{g}^{\,e}_{b} - 2\Omega^i_{ie} \vec{v}^{\,e}_{eb} \\ C^e_b \Omega^b_{eb} \end{bmatrix} \tag{7}$$

or

$$\begin{bmatrix} \dot{\vec{r}}^{\,e}_{eb} \\ \dot{\vec{v}}^{\,e}_{eb} \\ \dot{\bar{q}}^e_b \end{bmatrix} = \begin{bmatrix} \vec{v}^{\,e}_{eb} \\ C^e_b \vec{f}^{\,b}_{ib} + \vec{g}^{\,e}_{b} - 2\Omega^i_{ie} \vec{v}^{\,e}_{eb} \\ \frac{1}{2}[\breve{\omega}^b_{eb} \circledast] \bar{q}^{\,e}_{b}(t) \end{bmatrix} \tag{8}$$

where $\vec{\omega}^{\,b}_{ib} = \vec{\omega}^{\,b}_{ie} + \vec{\omega}^{\,b}_{eb}$, and $\Omega^b_{eb} = \Omega^b_{ib} - \Omega^b_{ie}$. .11

Appendix

$$[\bar{q}\otimes] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & -q_z & q_y \\ q_y & q_z & q_s & -q_x \\ q_z & -q_y & q_x & q_s \end{bmatrix}$$

$$[\bar{q}\circledast] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & q_z & -q_y \\ q_y & -q_z & q_s & q_x \\ q_z & q_y & -q_x & q_s \end{bmatrix}$$

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