# EE 565: Position, Navigation and Timing

## Navigation Equations: ECEF Mechanization

Aly El-Osery    Kevin Wedeward

Electrical Engineering Department, New Mexico Tech
Socorro, New Mexico, USA

*In Collaboration with*
Stephen Bruder
Electrical and Computer Engineering Department
Embry-Riddle Aeronautical Univesity
Prescott, Arizona, USA

February 22, 2018

- Determine the position, velocity and attitude of the **body** frame *wrt* the **Earth** frame.
  - **Position** — Vector from the origin of the earth frame to the origin of the body frame resolved in the earth frame: $\vec{r}^{\,e}_{eb}$
  - **Velocity** — Velocity of the body frame *wrt* the earth frame resolved in the earth frame: $\vec{v}^{\,e}_{eb}$
  - **Attitude** — Orientation of the body frame *wrt* the earth frame: $C^e_b$

- Body orientation frame at time "$k$" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$
- Start with angular velocity

$$\vec{\omega}_{ib}^{e} =$$

- Body orientation frame at time "$k$" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$
- Start with angular velocity

$$\vec{\omega}^{\,e}_{ib} = \vec{\omega}^{\,e}_{ie} + \vec{\omega}^{\,e}_{eb}$$

$$\vec{\omega}^{\,e}_{eb} =$$

- Body orientation frame at time "$k$" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$
- Start with angular velocity

$$\vec{\omega}^{\,e}_{ib} = \vec{\omega}^{\,e}_{ie} + \vec{\omega}^{\,e}_{eb}$$
$$\vec{\omega}^{\,e}_{eb} = C^e_b \vec{\omega}^{\,b}_{ib} - \vec{\omega}^{\,e}_{ie}$$

- Body orientation frame at time "$k$" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$
- Start with angular velocity

$$\vec{\omega}^{\,e}_{ib} = \vec{\omega}^{\,e}_{ie} + \vec{\omega}^{\,e}_{eb}$$

$$\vec{\omega}^{\,e}_{eb} = C^e_b \vec{\omega}^{\,b}_{ib} - \vec{\omega}^{\,e}_{ie}$$

$$\Omega^e_{eb} = C^e_b \Omega^b_{ib} C^b_e - \Omega^e_{ie}$$

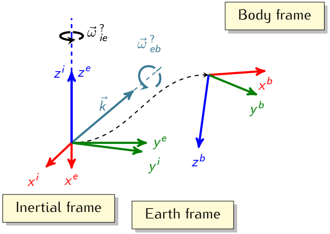
- Body orientation frame at time “$k$” *wrt* time “$k-1$”
  - $\Delta t = t_k - t_{k-1}$
- Start with angular velocity

$$\vec{\omega}^{\,e}_{ib} = \vec{\omega}^{\,e}_{ie} + \vec{\omega}^{\,e}_{eb}$$

$$\vec{\omega}^{\,e}_{eb} = C^e_b \vec{\omega}^{\,b}_{ib} - \vec{\omega}^{\,e}_{ie}$$

$$\Omega^e_{eb} = C^e_b \Omega^b_{ib} C^b_e - \Omega^e_{ie}$$

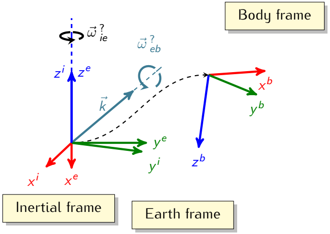
$$C^e_b(+) - C^e_b(-) \approx$$



- Body orientation frame at time "$k$" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$
- Start with angular velocity

$$\vec{\omega}^{\,e}_{ib} = \vec{\omega}^{\,e}_{ie} + \vec{\omega}^{\,e}_{eb}$$
$$\vec{\omega}^{\,e}_{eb} = C^e_b \vec{\omega}^{\,b}_{ib} - \vec{\omega}^{\,e}_{ie}$$
$$\Omega^e_{eb} = C^e_b \Omega^b_{ib} C^b_e - \Omega^e_{ie}$$

$$C^e_b(+) - C^e_b(-) \approx \Delta t \Omega^e_{eb} C^e_b(-)$$

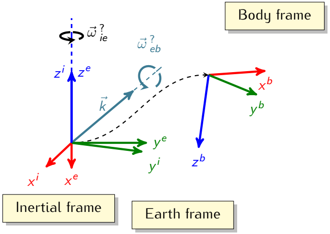


- Body orientation frame at time "$k$" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$
- Start with angular velocity

$$\vec{\omega}^{\,e}_{ib} = \vec{\omega}^{\,e}_{ie} + \vec{\omega}^{\,e}_{eb}$$
$$\vec{\omega}^{\,e}_{eb} = C^e_b \vec{\omega}^{\,b}_{ib} - \vec{\omega}^{\,e}_{ie}$$
$$\Omega^e_{eb} = C^e_b \Omega^b_{ib} C^b_e - \Omega^e_{ie}$$

$$C^e_b(+) - C^e_b(-) \approx \Delta t \Omega^e_{eb} C^e_b(-)$$

$$C^e_b(+) \approx C^e_b(-) + \Delta t \left( C^e_b \Omega^b_{ib} C^b_e - \Omega^e_{ie} \right) C^e_b(-)$$
$$= C^e_b(-) \left( \mathcal{I} + \Omega^b_{ib} \Delta t \right) - \Omega^e_{ie} C^e_b(-) \Delta t$$



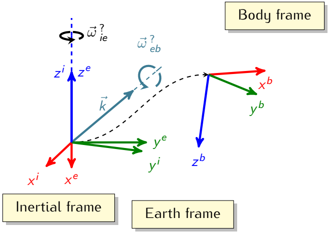
- Body orientation frame at time "$k$" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$
- Start with the angular velocity

$$\Omega_{eb}^{e} = C_b^e \Omega_{ib}^{b} C_e^b - \Omega_{ie}^{e}$$

$$C_b^e(+) = C_b^e(-) e^{\Omega_{eb}^{b}\Delta t} = e^{\Omega_{eb}^{e}\Delta t} C_b^e(-)$$

$$C_b^e(+) = \left[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1-\cos(\Delta\theta)]\,\mathfrak{K}^2\right] C_b^e(-)$$

$$e^{\Omega_{eb}^{e}\Delta t} = e^{\mathfrak{K}\theta}$$



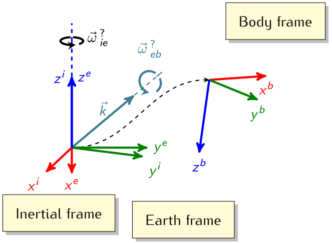
- Body orientation frame at time "$k$" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$

$$\vec{\omega}^{\,e}_{eb}\Delta t = \vec{k}\Delta\theta$$

$$\bar{q}^{\,e}_{\,b}(+) = \Delta\bar{q}^{\,e}_{\,b} \otimes \bar{q}^{\,e}_{\,b}(-)$$

$$\Delta\bar{q}^{\,e}_{\,b} = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix}$$

Need to periodically renormalize $\bar{q}$



NEW MEXICO TECH
SCIENCE • ENGINEERING • RESEARCH UNIVERSITY

$\vec{\omega}\,^e_{eb}\Delta t = \vec{k}\Delta\theta$ $\mathfrak{K} = [\vec{k}\times]$

- High fidelity

$$C^e_b(+) = \left[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1 - \cos(\Delta\theta)]\,\mathfrak{K}^2\right]\,C^e_b(-) \quad (1)$$

or

$$\bar{q}\,^e_b(+) = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix} \otimes \bar{q}\,^e_b(-) \quad (2)$$

- Low fidelity

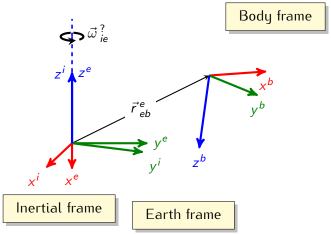
$$C^e_b(+) \approx C^e_b(-)\left(\mathcal{I} + \Omega^b_{ib}\Delta t\right) - \Omega^e_{ie}C^e_b(-)\Delta t \quad (3)$$

2. Specific force transformation
   - Simply coordinatize the specific force

$$\vec{f}^{\,e}_{ib} = C^e_b(+)\vec{f}^{\,b}_{ib} \tag{4}$$

3. Velocity update

$$\vec{r}^{\,i}_{ib} =$$

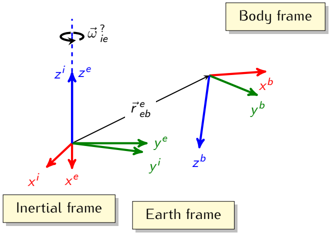


2. Specific force transformation
   - Simply coordinatize the specific force

$$\vec{f}^{\,e}_{ib} = C^e_b(+)\vec{f}^{\,b}_{ib} \quad (4)$$

3. Velocity update

$$\vec{r}^{\,i}_{ib} = \cancelto{0}{\vec{r}^{\,i}_{ie}} + C^i_e \vec{r}^{\,e}_{eb} \Rightarrow \vec{r}^{\,e}_{eb} = C^e_i \vec{r}^{\,i}_{ib}$$



NEW MEXICO TECH
SCIENCE • ENGINEERING • RESEARCH UNIVERSITY

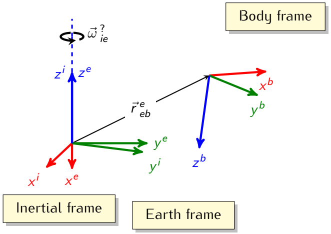
2. Specific force transformation
   - Simply coordinatize the specific force

$$\vec{f}^{\,e}_{ib} = C^e_b(+)\vec{f}^{\,b}_{ib} \tag{4}$$

3. Velocity update

$$\vec{r}^{\,i}_{ib} = \overset{0}{\cancel{\vec{r}^{\,i}_{ie}}} + C^i_e \vec{r}^{\,e}_{eb} \Rightarrow \vec{r}^{\,e}_{eb} = C^e_i \vec{r}^{\,i}_{ib}$$

$$\vec{v}^{\,e}_{eb} =$$



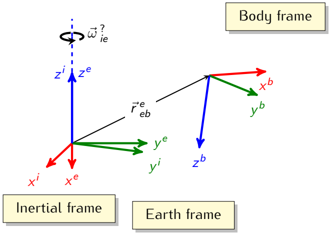
1. Specific force transformation
   - Simply coordinatize the specific force

$$\vec{f}^{\,e}_{ib} = C^e_b(+)\vec{f}^{\,b}_{ib} \tag{4}$$

2. Velocity update

$$\vec{r}^{\,i}_{ib} = \cancel{\vec{r}^{\,i}_{ie}}^{\,0} + C^i_e\vec{r}^{\,e}_{eb} \Rightarrow \vec{r}^{\,e}_{eb} = C^e_i\vec{r}^{\,i}_{ib}$$

$$\begin{aligned}\vec{v}^{\,e}_{eb} &= \dot{\vec{r}}^{\,e}_{eb} \\ &= \dot{C}^e_i\vec{r}^{\,i}_{ib} + C^e_i\dot{\vec{r}}^{\,i}_{ib} \\ &= \Omega^e_{ei}C^e_i\vec{r}^{\,i}_{ib} + C^e_i\vec{v}^{\,i}_{ib} \\ &= -\Omega^e_{ie}\vec{r}^{\,e}_{eb} + C^e_i\vec{v}^{\,i}_{ib}\end{aligned}$$



$$
\begin{aligned}
\vec{a}^{\,e}_{eb} = \dot{\vec{v}}^{\,e}_{eb} &= \frac{d}{dt}\left(-\Omega^e_{ie}\vec{r}^{\,e}_{eb} + C^e_i \vec{v}^{\,i}_{ib}\right) \\
&= -\Omega^e_{ie}\dot{\vec{r}}^{\,e}_{eb} + \dot{C}^e_i \vec{v}^{\,i}_{ib} + C^e_i \dot{\vec{v}}^{\,i}_{ib} \\
&=
\end{aligned}
$$

$$\vec{a}^{\,e}_{eb} = \dot{\vec{v}}^{\,e}_{eb} = \frac{d}{dt}\left(-\Omega^e_{ie}\vec{r}^{\,e}_{eb} + C^e_i \vec{v}^{\,i}_{ib}\right)$$

$$= -\Omega^e_{ie}\dot{\vec{r}}^{\,e}_{eb} + \dot{C}^e_i \vec{v}^{\,i}_{ib} + C^e_i \dot{\vec{v}}^{\,i}_{ib}$$

$$= -\Omega^e_{ie}\vec{v}^{\,e}_{eb} + \Omega^e_{ei}\, C^e_i \vec{v}^{\,i}_{ib} + C^e_i \vec{a}^{\,i}_{ib}$$

$$\vec{a}^{\,e}_{eb} = \dot{\vec{v}}^{\,e}_{eb} = \frac{d}{dt}\left(-\Omega^e_{ie}\vec{r}^{\,e}_{eb} + C^e_i\vec{v}^{\,i}_{ib}\right)$$

$$= -\Omega^e_{ie}\dot{\vec{r}}^{\,e}_{eb} + \dot{C}^e_i\vec{v}^{\,i}_{ib} + C^e_i\dot{\vec{v}}^{\,i}_{ib}$$

$$= -\Omega^e_{ie}\vec{v}^{\,e}_{eb} + \Omega^e_{ei}C^e_i\vec{v}^{\,i}_{ib} + C^e_i\vec{a}^{\,i}_{ib}$$

$$C^e_i\vec{v}^{\,i}_{ib} = \Omega^e_{ie}\vec{r}^{\,e}_{eb} + \vec{v}^{\,e}_{eb}$$

# Steps 2–4

$$
\begin{aligned}
\vec{a}^{\,e}_{eb} = \dot{\vec{v}}^{\,e}_{eb} &= \frac{d}{dt}\left(-\Omega^e_{ie}\vec{r}^{\,e}_{eb} + C^e_i \vec{v}^{\,i}_{ib}\right) \\
&= -\Omega^e_{ie}\dot{\vec{r}}^{\,e}_{eb} + \dot{C}^e_i \vec{v}^{\,i}_{ib} + C^e_i \dot{\vec{v}}^{\,i}_{ib} \\
&= -\Omega^e_{ie}\vec{v}^{\,e}_{eb} + \Omega^e_{ei} C^e_i \vec{v}^{\,i}_{ib} + C^e_i \vec{a}^{\,i}_{ib} \\
&= -\Omega^e_{ie}\vec{v}^{\,e}_{eb} - \Omega^e_{ie}\left[\vec{v}^{\,e}_{eb} + \Omega^e_{ie}\vec{r}^{\,e}_{eb}\right] + C^e_i \vec{a}^{\,i}_{ib} \\
&= -2\Omega^e_{ie}\vec{v}^{\,e}_{eb} - \Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb} + \vec{a}^{\,e}_{ib}
\end{aligned}
$$

$$C^e_i \vec{v}^{\,i}_{ib} = \Omega^e_{ie}\vec{r}^{\,e}_{eb} + \vec{v}^{\,e}_{eb}$$

$$\vec{a}^{\,e}_{eb} = \dot{\vec{v}}^{\,e}_{eb} = \frac{d}{dt}\left(-\Omega^e_{ie}\vec{r}^{\,e}_{eb} + C^e_i \vec{v}^{\,i}_{ib}\right)$$
$$= -\Omega^e_{ie}\dot{\vec{r}}^{\,e}_{eb} + \dot{C}^e_i \vec{v}^{\,i}_{ib} + C^e_i \dot{\vec{v}}^{\,i}_{ib}$$
$$= -\Omega^e_{ie}\vec{v}^{\,e}_{eb} + \Omega^e_{ei} C^e_i \vec{v}^{\,i}_{ib} + C^e_i \vec{a}^{\,i}_{ib}$$
$$= -\Omega^e_{ie}\vec{v}^{\,e}_{eb} - \Omega^e_{ie}\left[\vec{v}^{\,e}_{eb} + \Omega^e_{ie}\vec{r}^{\,e}_{eb}\right] + C^e_i \vec{a}^{\,i}_{ib}$$
$$= -2\Omega^e_{ie}\vec{v}^{\,e}_{eb} - \Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb} + \vec{a}^{\,e}_{ib}$$

$$C^e_i \vec{v}^{\,i}_{ib} = \Omega^e_{ie}\vec{r}^{\,e}_{eb} + \vec{v}^{\,e}_{eb}$$

$$\vec{f}^{\,?}_{ib} = \vec{a}^{\,?}_{ib} - \vec{\gamma}^{\,?}_{ib}$$

$$\vec{a}^{\,e}_{ib} = \vec{f}^{\,e}_{ib} + \vec{\gamma}^{\,e}_{ib}$$

$$\vec{g}^{\,e}_{b} = \vec{\gamma}^{\,e}_{ib} - \Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb}$$

$$\vec{a}^{\,e}_{ib} = \vec{f}^{\,e}_{ib} + \vec{g}^{\,e}_{b} + \Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb}$$

$$
\begin{aligned}
\vec{a}^{\,e}_{eb} = \dot{\vec{v}}^{\,e}_{eb} &= \frac{d}{dt}\left(-\Omega^e_{ie}\vec{r}^{\,e}_{eb} + C^e_i \vec{v}^{\,i}_{ib}\right) \\
&= -\Omega^e_{ie}\dot{\vec{r}}^{\,e}_{eb} + \dot{C}^e_i \vec{v}^{\,i}_{ib} + C^e_i \dot{\vec{v}}^{\,i}_{ib} \\
&= -\Omega^e_{ie}\vec{v}^{\,e}_{eb} + \Omega^e_{ei} C^e_i \vec{v}^{\,i}_{ib} + C^e_i \vec{a}^{\,i}_{ib} \\
&= -\Omega^e_{ie}\vec{v}^{\,e}_{eb} - \Omega^e_{ie}\left[\vec{v}^{\,e}_{eb} + \Omega^e_{ie}\vec{r}^{\,e}_{eb}\right] + C^e_i \vec{a}^{\,i}_{ib} \\
&= -2\Omega^e_{ie}\vec{v}^{\,e}_{eb} - \Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb} + \vec{a}^{\,e}_{ib} \\
&= -2\Omega^e_{ie}\vec{v}^{\,e}_{eb} + \vec{f}^{\,e}_{ib} + \vec{g}^{\,e}_{b}
\end{aligned}
$$

$$C^e_i \vec{v}^{\,i}_{ib} = \Omega^e_{ie}\vec{r}^{\,e}_{eb} + \vec{v}^{\,e}_{eb}$$

$$\vec{f}^{\,?}_{ib} = \vec{a}^{\,?}_{ib} - \vec{\gamma}^{\,?}_{ib}$$

$$\vec{a}^{\,e}_{ib} = \vec{f}^{\,e}_{ib} + \vec{\gamma}^{\,e}_{ib}$$

$$\vec{g}^{\,e}_{b} = \vec{\gamma}^{\,e}_{ib} - \Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb}$$

$$\vec{a}^{\,e}_{ib} = \vec{f}^{\,e}_{ib} + \vec{g}^{\,e}_{b} + \Omega^e_{ie}\Omega^e_{ie}\vec{r}^{\,e}_{eb}$$

$$\vec{a}^{e}_{eb} = \dot{\vec{v}}^{e}_{eb} = \frac{d}{dt}\left(-\Omega^e_{ie}\vec{r}^{e}_{eb} + C^e_i\vec{v}^{i}_{ib}\right)$$
$$= -\Omega^e_{ie}\dot{\vec{r}}^{e}_{eb} + \dot{C}^e_i\vec{v}^{i}_{ib} + C^e_i\dot{\vec{v}}^{i}_{ib}$$
$$= -\Omega^e_{ie}\vec{v}^{e}_{eb} + \Omega^e_{ei} C^e_i\vec{v}^{i}_{ib} + C^e_i\vec{a}^{i}_{ib}$$
$$= -\Omega^e_{ie}\vec{v}^{e}_{eb} - \Omega^e_{ie}\left[\vec{v}^{e}_{eb} + \Omega^e_{ie}\vec{r}^{e}_{eb}\right] + C^e_i\vec{a}^{i}_{ib}$$
$$= -2\Omega^e_{ie}\vec{v}^{e}_{eb} - \Omega^e_{ie}\Omega^e_{ie}\vec{r}^{e}_{eb} + \vec{a}^{e}_{ib}$$
$$= -2\Omega^e_{ie}\vec{v}^{e}_{eb} + \vec{f}^{e}_{ib} + \vec{g}^{e}_{b}$$

$$C^e_i\vec{v}^{i}_{ib} = \Omega^e_{ie}\vec{r}^{e}_{eb} + \vec{v}^{e}_{eb}$$

$$\vec{f}^{?}_{ib} = \vec{a}^{?}_{ib} - \vec{\gamma}^{?}_{ib}$$

$$\vec{a}^{e}_{ib} = \vec{f}^{e}_{ib} + \vec{\gamma}^{e}_{ib}$$

$$\vec{g}^{e}_{b} = \vec{\gamma}^{e}_{ib} - \Omega^e_{ie}\Omega^e_{ie}\vec{r}^{e}_{eb}$$

$$\vec{a}^{e}_{ib} = \vec{f}^{e}_{ib} + \vec{g}^{e}_{b} + \Omega^e_{ie}\Omega^e_{ie}\vec{r}^{e}_{eb}$$

$$\vec{v}^{e}_{eb}(+) = \vec{v}^{e}_{eb}(-) + \vec{a}^{e}_{eb}\Delta t$$
$$= \vec{v}^{e}_{eb}(-) + \left[\vec{f}^{e}_{ib} + \vec{g}^{e}_{b} - 2\Omega^i_{ie}\vec{v}^{e}_{eb}(-)\right]\Delta t \qquad (5)$$

4. Position update
   - by simple numerical integration

$$\vec{r}^{\,e}_{eb}(+) = \vec{r}^{\,e}_{eb}(-) + \vec{v}^{\,e}_{eb}(-)\Delta t + \vec{a}^{\,e}_{eb}\frac{\Delta t^2}{2} \tag{6}$$

$$C_b^e(+) = \left[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1 - \cos(\Delta\theta)]\,\mathfrak{K}^2\right] C_b^e(-)$$

or

$$C_b^e(+) \approx C_b^e(-)\left(\mathcal{I} + \Omega_{ib}^b \Delta t\right) - \Omega_{ie}^i C_b^e(-)\Delta t$$

or

$$\bar{q}\,_b^e(+) = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix} \otimes \bar{q}\,_b^e(-)$$

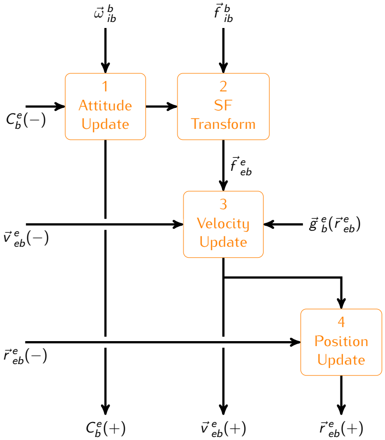
and

$$\vec{f}\,_{ib}^e = C_b^e(+)\vec{f}\,_{ib}^b$$

$$\vec{a}\,_{eb}^e = \vec{f}\,_{ib}^e + \vec{g}\,_b^e - 2\Omega_{ie}^i \vec{v}\,_{eb}^e(-)$$

$$\vec{v}\,_{eb}^e(+) = \vec{v}\,_{eb}^e(-) + \vec{a}\,_{eb}^e \Delta t$$

$$\vec{r}\,_{eb}^e(+) = \vec{r}\,_{eb}^e(-) + \vec{v}\,_{eb}^e(-)\Delta t + \vec{a}\,_{eb}^e \frac{\Delta t^2}{2}$$



- In continuous time notation
  - Attitude: $\dot{C}_b^e = C_b^e \Omega_{eb}^b$ or $\dot{\bar{q}}_b^e = \frac{1}{2}[\breve{\omega}_{eb}^b \circledast]\bar{q}\,_b^e(t)$
  - Velocity: $\dot{\vec{v}}\,_{eb}^e = C_b^e \vec{f}\,_{ib}^b + \vec{g}\,_b^e - 2\Omega_{ie}^i \vec{v}\,_{eb}^e$
  - Position: $\dot{\vec{r}}\,_{ib}^e = \vec{v}\,_{eb}^e$

$$\vec{\omega}\,_{ib}^b = \vec{\omega}\,_{ie}^b + \vec{\omega}\,_{eb}^b$$

$$\Omega_{eb}^b = \Omega_{ib}^b - \Omega_{ie}^b$$

- In State-space notation

$$\begin{bmatrix} \dot{\vec{r}}\,_{eb}^e \\ \dot{\vec{v}}\,_{eb}^e \\ \dot{C}_b^i \end{bmatrix} = \begin{bmatrix} \vec{v}\,_{eb}^e \\ C_b^e \vec{f}\,_{ib}^b + \vec{g}\,_b^e - 2\Omega_{ie}^i \vec{v}\,_{eb}^e \\ C_b^e \Omega_{eb}^b \end{bmatrix} \tag{7}$$

or

$$\begin{bmatrix} \dot{\vec{r}}\,_{eb}^e \\ \dot{\vec{v}}\,_{eb}^e \\ \dot{\bar{q}}_b^e \end{bmatrix} = \begin{bmatrix} \vec{v}\,_{eb}^e \\ C_b^e \vec{f}\,_{ib}^b + \vec{g}\,_b^e - 2\Omega_{ie}^i \vec{v}\,_{eb}^e \\ \frac{1}{2}[\breve{\omega}_{eb}^b \circledast]\bar{q}\,_b^e(t) \end{bmatrix} \tag{8}$$

$$[\bar{q}\otimes] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & -q_z & q_y \\ q_y & q_z & q_s & -q_x \\ q_z & -q_y & q_x & q_s \end{bmatrix}$$

$$[\bar{q}\circledast] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & q_z & -q_y \\ q_y & -q_z & q_s & q_x \\ q_z & q_y & -q_x & q_s \end{bmatrix}$$