

# Lecture

## Navigation Equations: Nav Mechanization

EE 565: Position, Navigation and Timing

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### Nav Mechanization

- Determine the position, velocity and attitude of the **body** frame *wrt* the **Nav** frame.
  - **Position** — Typically described in curvilinear coordinates:  $[L_b, \lambda_b, h_b]^T$
  - **Velocity** — Velocity of the body frame *wrt* the earth frame resolved in the navigation frame:  $\vec{v}_{eb}^n$
  - **Attitude** — Orientation of the body frame *wrt* the navigation frame:  $C_b^n$

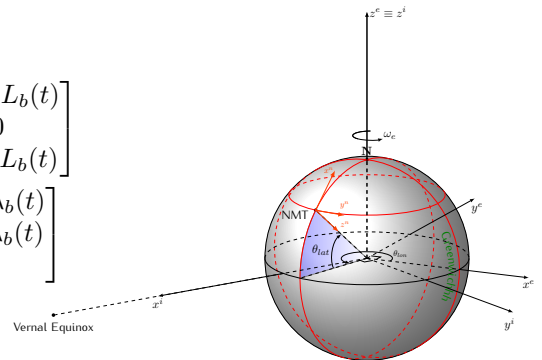
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### ECEF/Nav

- Description of the Nav frame
  - Orientation of the  $n$ -frame *wrt* the  $e$ -frame

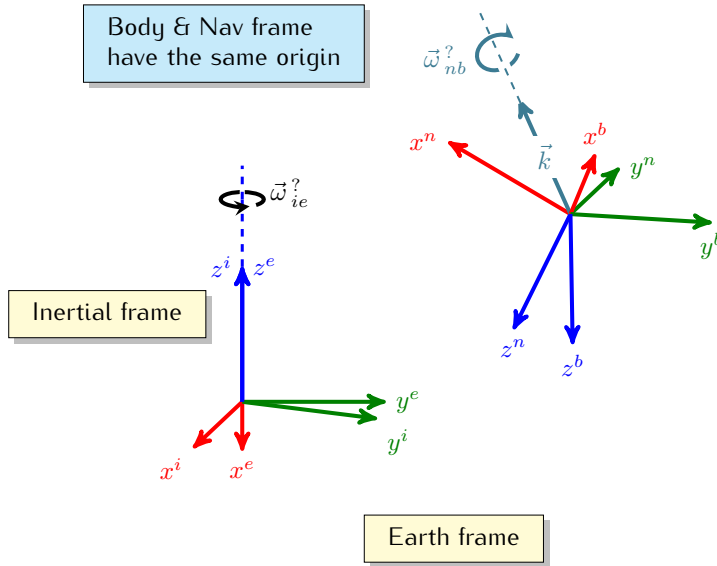
$$\begin{aligned}
 C_n^e(t) &= R_{(z, \lambda_b(t))} R_{(\vec{y}, -L_b(t) - 90^\circ)} \\
 &= \begin{bmatrix} \cos \lambda_b(t) & -\sin \lambda_b(t) & 0 \\ \sin \lambda_b(t) & \cos \lambda_b(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin L_b(t) & 0 & -\cos L_b(t) \\ 0 & 1 & 0 \\ \cos L_b(t) & 0 & -\sin L_b(t) \end{bmatrix} \\
 &= \begin{bmatrix} -\sin L_b(t) \cos \lambda_b(t) & -\sin \lambda_b(t) & -\cos L_b(t) \cos \lambda_b(t) \\ -\sin L_b(t) \sin \lambda_b(t) & \cos \lambda_b(t) & -\cos L_b(t) \sin \lambda_b(t) \\ \cos L_b(t) & 0 & -\sin L_b(t) \end{bmatrix}
 \end{aligned}$$

where geodetic Lat =  $L_b$   
and Geodetic Lon =  $\lambda_b$



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## Body wrt Nav Frame



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### Attitude — Method A

- Start with angular velocity

$$\vec{\omega}_{ib}^b = \vec{\omega}_{ie}^b + \vec{\omega}_{en}^b + \vec{\omega}_{nb}^b \rightarrow \vec{\omega}_{nb}^b = \vec{\omega}_{ib}^b - \vec{\omega}_{ie}^b - \vec{\omega}_{en}^b$$

- Now

$$\begin{aligned} \dot{C}_b^n &= C_b^n \Omega_{nb}^b = C_b^n (\Omega_{ib}^b - \Omega_{ie}^b - \Omega_{en}^b) \\ &= C_b^n \Omega_{ib}^b - C_b^n \Omega_{ie}^b - C_b^n \Omega_{en}^b \\ &= C_b^n \Omega_{ib}^b - (\Omega_{ie}^n + \Omega_{en}^n) C_b^n \end{aligned}$$

Recall that:

$$\begin{aligned} [(C\vec{\omega}) \times] &= C[\vec{\omega} \times] C^T \\ C[\vec{\omega} \times] &= [(C\vec{\omega}) \times] C \end{aligned}$$

$$\omega_{ie}^n = C_e^n \omega_{ie}^e = \omega_{ie} \begin{bmatrix} \cos L_b \\ 0 \\ -\sin L_b \end{bmatrix}$$

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### Attitude — Method A

- $\Omega_{en}^n = [\vec{\omega}_{en}^n \times]$
- Courtesy of Prof. Bruder and Mathematica

$$\dot{C}_n^e = C_n^e \Omega_{en}^n \rightarrow \Omega_{en}^n = (C_n^e)^T \dot{C}_n^e = \begin{bmatrix} 0 & \dot{\lambda}_b \sin L_b & -\dot{L}_b \\ -\dot{\lambda}_b \sin L_b & 0 & -\dot{\lambda}_b \cos L_b \\ \dot{L}_b & \dot{\lambda}_b \cos L_b & 0 \end{bmatrix}$$

- therefore,

$$\omega_{en}^n = \begin{bmatrix} \dot{\lambda}_b \cos L_b \\ -\dot{L}_b \\ -\dot{\lambda}_b \sin L_b \end{bmatrix}$$

- Finally since,

- then

$$\begin{bmatrix} \dot{L}_b \\ \dot{\lambda}_b \\ \dot{h}_b \end{bmatrix} = \begin{bmatrix} \frac{\vec{v}_{eb,N}^n}{R_N + h_b} \\ \frac{\vec{v}_{eb,E}^n}{\cos(L_b)(R_E + h_b)} \\ -\vec{v}_{eb,D}^n \end{bmatrix} \quad \omega_{en}^n = \begin{bmatrix} \frac{\vec{v}_{eb,E}^n}{R_E + h_b} \\ -\frac{\vec{v}_{eb,N}^n}{R_N + h_b} \\ -\frac{\tan(L_b) \vec{v}_{eb,E}^n}{R_E + h_b} \end{bmatrix} \quad (1)$$

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## Attitude — Method A

$$\begin{aligned}
 C_b^m(+)-C_b^m(-) &\approx \Delta t \dot{C}_b^m \\
 C_b^m(+)&\approx C_b^m(-)+\Delta t\left[C_b^n \Omega_{ib}^b-\left(\Omega_{ie}^n+\Omega_{en}^n\right) C_b^m(-)\right] \\
 &= C_b^m(-)\left(\mathcal{I}+\Omega_{ib}^b \Delta t\right)-\left(\Omega_{ie}^n+\Omega_{en}^n\right) C_b^m(-) \Delta t
 \end{aligned}$$

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## Attitude — Method B

- Body orientation frame at time “ $k$ ” wrt time “ $k-1$ ”

$$-\Delta t=t_k-t_{k-1}$$

- Start with the angular velocity

$$\begin{aligned}
 \omega_{nb}^b &= \omega_{ib}^b-\omega_{ie}^b-\omega_{en}^b \\
 \Omega_{nb}^b &= \Omega_{ib}^b-\Omega_{ie}^b-\Omega_{en}^b \\
 &= \Omega_{ib}^b-C_n^b \Omega_{ie}^n C_b^m-C_n^b \Omega_{en}^n C_b^m \\
 C_b^m(+)&= C_b^m(-) e^{\Omega_{nb}^b \Delta t} \\
 C_b^m(+)&= C_b^m(-)\left[\mathcal{I}+\sin(\Delta \theta) \mathfrak{K}+[1-\cos(\Delta \theta)] \mathfrak{K}^2\right]
 \end{aligned}$$

$$e^{\Omega_{nb}^b \Delta t}=e^{\mathfrak{K} \Delta \theta}$$

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## Attitude — Method C

- Body orientation frame at time “ $k$ ” wrt time “ $k-1$ ”

$$-\Delta t=t_k-t_{k-1}$$

$$\bar{q}_b^n(+)=\bar{q}_b^n(-) \otimes \Delta \bar{q}_{b(k)}^{b(k-1)} \quad \vec{\omega}_{nb}^b \Delta t=\vec{k} \Delta \theta$$

$$\Delta \bar{q}_{b(k)}^{b(k-1)}=\left[\begin{array}{c} \cos\left(\frac{\Delta \theta}{2}\right) \\ \vec{k} \sin\left(\frac{\Delta \theta}{2}\right) \end{array}\right]$$

Need to periodically renormalize  $\bar{q}$

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## Attitude Update— Summary

- High fidelity

$$C_b^m(+)=C_b^m(-)\left[\mathcal{I}+\sin(\Delta \theta) \mathfrak{K}+[1-\cos(\Delta \theta)] \mathfrak{K}^2\right] \quad (2)$$

or

$$\begin{aligned}
 \bar{q}_b^n(+)&= \bar{q}_b^n(-) \otimes \Delta \bar{q}_{b(k)}^{b(k-1)} \\
 \Delta \bar{q}_{b(k)}^{b(k-1)} &= \left[\begin{array}{c} \cos\left(\frac{\Delta \theta}{2}\right) \\ \vec{k} \sin\left(\frac{\Delta \theta}{2}\right) \end{array}\right] \quad (3)
 \end{aligned}$$

- Low fidelity

$$C_b^m(+)\approx C_b^m(-)\left(\mathcal{I}+\Omega_{ib}^b \Delta t\right)-\left(\Omega_{ie}^n+\Omega_{en}^n\right) C_b^m(-) \Delta t \quad (4)$$

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## Steps 2–4

### 2. Specific force transformation

- Simply coordinatize the specific force

$$\vec{f}_{ib}^n=C_b^m(+)\vec{f}_{ib}^b \quad (5)$$

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Steps 2–4

3. Velocity update

$$\dot{\vec{v}}_{eb}^e = \vec{a}_{eb}^e = \vec{f}_{ib}^e + \vec{g}_b^e - 2\Omega_{ie}^e \vec{v}_{eb}^e$$

- Note that:  $\vec{v}_{eb}^n = C_e^n \vec{v}_{eb}^e$

$$\begin{aligned} \dot{\vec{v}}_{eb}^n &= \dot{C}_e^n \vec{v}_{eb}^e + C_e^n \dot{\vec{v}}_{eb}^e \\ &= \Omega_{ne}^n C_e^n \vec{v}_{eb}^e + C_e^n \left( \vec{f}_{ib}^e + \vec{g}_b^e - 2\Omega_{ie}^e \vec{v}_{eb}^e \right) \\ &= \vec{f}_{ib}^n + \vec{g}_b^n - \Omega_{en}^n \vec{v}_{eb}^n - 2C_e^n \Omega_{ie}^e \vec{v}_{eb}^e \\ &= \vec{f}_{ib}^n + \vec{g}_b^n - \Omega_{en}^n \vec{v}_{eb}^n - 2\Omega_{ie}^n C_e^n \vec{v}_{eb}^e \\ &= \vec{f}_{ib}^n + \vec{g}_b^n - (\Omega_{en}^n + 2\Omega_{ie}^n) \vec{v}_{eb}^n \end{aligned}$$

- Finally,

$$\vec{v}_{eb}^n(+) = \vec{v}_{eb}^n(-) + \Delta t \left[ \vec{f}_{ib}^n + \vec{g}_b^n - (\Omega_{en}^n + 2\Omega_{ie}^n) \vec{v}_{eb}^n(-) \right]$$

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Steps 2–4

4. Position update

- Recalling that

$$\begin{bmatrix} \dot{L}_b \\ \dot{\lambda}_b \\ \dot{h}_b \end{bmatrix} = \begin{bmatrix} \frac{\vec{v}_{eb,N}^n}{R_N + h_b} \\ \frac{\vec{v}_{eb,E}^n}{\cos(L_b)(R_E + h_b)} \\ -\vec{v}_{eb,D}^n \end{bmatrix}$$

5. then

$$\begin{aligned} h_b(+) &= h_b(-) - \Delta t \left[ \vec{v}_{eb,D}^n \right] \\ L_b(+) &= L_b(-) + \Delta t \left[ \frac{\vec{v}_{eb,N}^n}{R_N + h_b} \right] \\ \lambda_b(+) &= \lambda_b(-) + \Delta t \left[ \frac{\vec{v}_{eb,E}^n}{\cos(L_b)(R_E + h_b)} \right] \end{aligned}$$

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## Nav Mechanization Summary

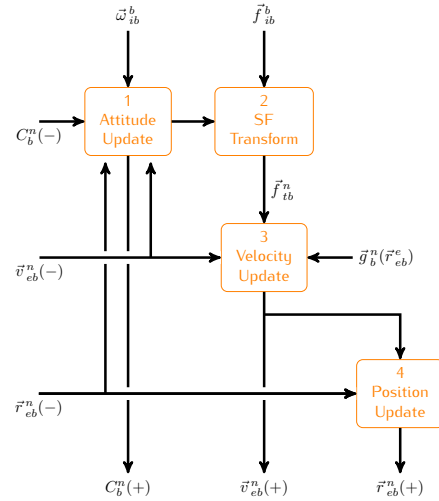
$$C_b^n(+) = C_b^n(-) [\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1 - \cos(\Delta\theta)]\mathfrak{K}^2] \quad (6)$$

or

$$\bar{q}_b^n(+) = \bar{q}_b^n(-) \otimes \Delta\bar{q}_{b(k)}^{b(k-1)}, \quad \Delta\bar{q}_{b(k)}^{b(k-1)} = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k} \sin(\frac{\Delta\theta}{2}) \end{bmatrix} \quad (7)$$

or

$$C_b^n(+) \approx C_b^n(-) (\mathcal{I} + \Omega_{ib}^b \Delta t) - (\Omega_{ie}^n + \Omega_{en}^n) C_b^n(-) \Delta t \quad (8)$$



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## Nav Mechanization Summary

$$\vec{f}_{ib}^n = C_b^n(+) \vec{f}_{ib}^b \quad (9)$$

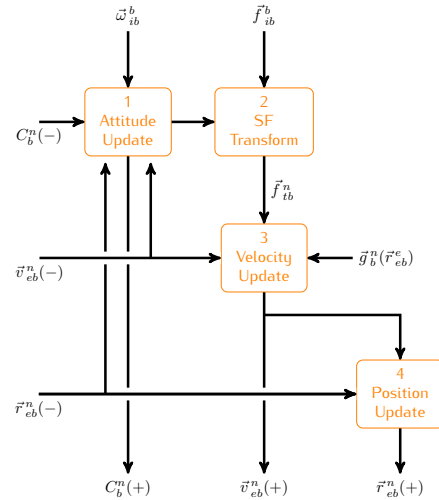
$$\vec{a}_{eb}^n = \vec{f}_{ib}^n + \vec{g}_b^n - (\Omega_{en}^n + 2\Omega_{ie}^n) \vec{v}_{eb}^n \quad (10)$$

$$\vec{v}_{eb}^n(+) = \vec{v}_{eb}^n(-) + \Delta t \left[ \vec{f}_{ib}^n + \vec{g}_b^n - (\Omega_{en}^n + 2\Omega_{ie}^n) \vec{v}_{eb}^n(-) \right] \quad (11)$$

$$L_b(+) = L_b(-) + \Delta t \left[ \frac{\vec{v}_{eb,N}^n}{R_N + h_b} \right] \quad (12)$$

$$\lambda_b(+) = \lambda_b(-) + \Delta t \left[ \frac{\vec{v}_{eb,E}^n}{\cos(L_b)(R_E + h_b)} \right] \quad (13)$$

$$h_b(+) = h_b(-) - \Delta t \left[ \vec{v}_{eb,D}^n \right] \quad (14)$$



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## Nav Mechanization — Continuous Case

- In continuous time notation

- Attitude:  $\dot{C}_b^n = C_b^n \Omega_{ib}^b - (\Omega_{ie}^n + \Omega_{en}^n) C_b^n$

- Velocity:  $\dot{\vec{v}}_{eb}^n = \vec{f}_{ib}^n + \vec{g}_b^n - (\Omega_{en}^n + 2\Omega_{ie}^n) \vec{v}_{eb}^n$

- Position:

$$\begin{bmatrix} \dot{L}_b \\ \dot{\lambda}_b \\ \dot{h}_b \end{bmatrix} = \begin{bmatrix} \frac{\vec{v}_{eb,N}^n}{R_N + h_b} \\ \frac{\vec{v}_{eb,E}^n}{\cos(L_b)(R_E + h_b)} \\ -\vec{v}_{eb,D}^n \end{bmatrix}$$

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## Nav Mechanization — Continuous Case

- In State-space notation

$$\begin{bmatrix} \dot{L}_b \\ \dot{\lambda}_b \\ \dot{h}_b \\ \dot{\vec{v}}_{eb}^n \\ \dot{C}_b^n \end{bmatrix} = \begin{bmatrix} \frac{\vec{v}_{eb,N}^n}{R_N+h_b} \\ \frac{\vec{v}_{eb,E}^n}{\cos(L_b)(R_E+h_b)} \\ -\vec{v}_{eb,D}^n \\ \vec{f}_{ib}^n + \vec{g}_b^n - (\Omega_{en}^n + 2\Omega_{ie}^n)\vec{v}_{eb}^n \\ C_b^n \Omega_{ib}^b - (\Omega_{ie}^n + \Omega_{en}^n)C_b^n \end{bmatrix} \quad (15)$$

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## Appendix

$$[\bar{q}^{\otimes}] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & -q_z & q_y \\ q_y & q_z & q_s & -q_x \\ q_z & -q_y & q_x & q_s \end{bmatrix}$$

$$[\bar{q}^{\circledast}] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & q_z & -q_y \\ q_y & -q_z & q_s & q_x \\ q_z & q_y & -q_x & q_s \end{bmatrix}$$

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