EE 565: Position, Navigation and Timing
Navigation Equations: Nav Mechanization

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- Determine the position, velocity and attitude of the **body** frame *wrt* the **Nav** frame.
    - **Position** — Typically described in curvlinear coordinates: $[L_b, \lambda_b, h_b]^T$
    - **Velocity** — Velocity of the body frame *wrt* the earth frame resolved in the navigation frame: $\vec{v}_{eb}^{\,n}$
    - **Attitude** — Orientation of the body frame *wrt* the navigation frame: C_b^n

- Description of the Nav frame
  - Orientation of the *n*-frame *wrt* the *e*-frame

$$C_n^e(t) = R_{(\vec{z}, \lambda_b(t))} R_{(\vec{y}, -L_b(t) - 90°)}$$

$$= \begin{bmatrix} \cos \lambda_b(t) & -\sin \lambda_b(t) & 0 \\ \sin \lambda_b(t) & \cos \lambda_b(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin L_b(t) & 0 & -\cos L_b(t) \\ 0 & 1 & 0 \\ \cos L_b(t) & 0 & -\sin L_b(t) \end{bmatrix}$$

$$= \begin{bmatrix} -\sin L_b(t) \cos \lambda_b(t) & -\sin \lambda_b(t) & -\cos L_b(t) \cos \lambda_b(t) \\ -\sin L_b(t) \sin \lambda_b(t) & \cos \lambda_b(t) & -\cos L_b(t) \sin \lambda_b(t) \\ \cos L_b(t) & 0 & -\sin L_b(t) \end{bmatrix}$$

where geodetic Lat $= L_b$ and Geodetic Lon $= \lambda_b$

Body & Nav frame have the same origin

Inertial frame

Earth frame

$\vec{\omega}_{nb}^{?}$

$\vec{\omega}_{ie}^{?}$

$\vec{k}$

x^n x^b y^n y^b z^n z^b

z^i z^e

y^e y^i

x^i x^e
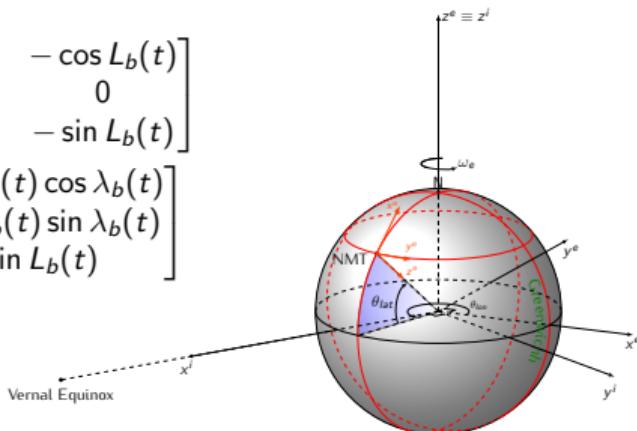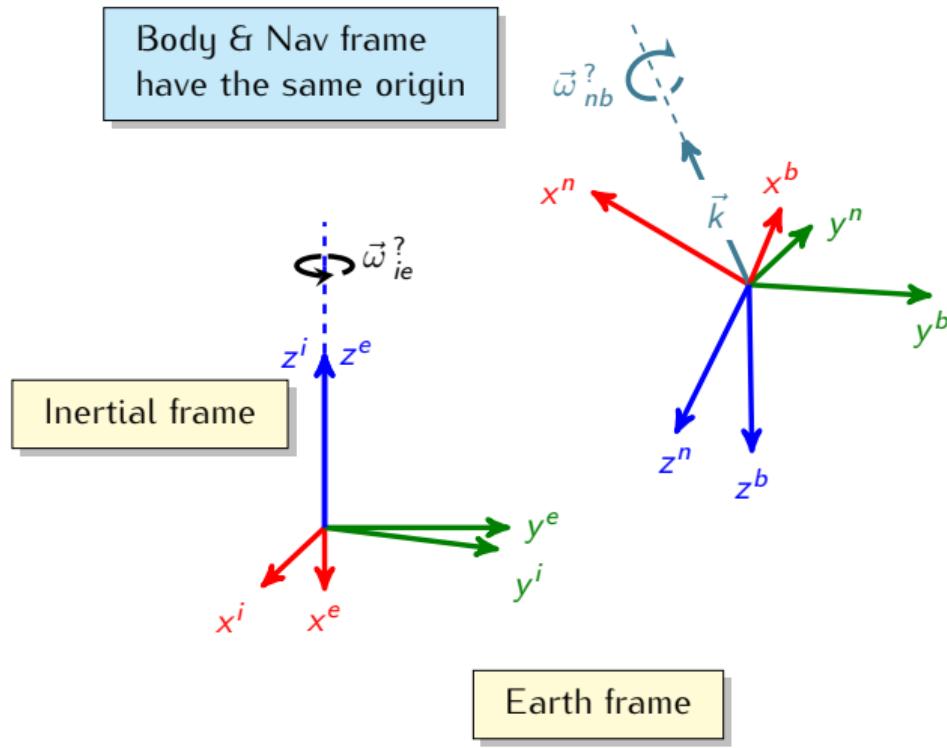
- Start with angular velocity

$$\vec{\omega}_{ib}^{b} =$$

- Start with angular velocity

$$\vec{\omega}_{ib}^{\,b} = \vec{\omega}_{ie}^{\,b} + \vec{\omega}_{en}^{\,b} + \vec{\omega}_{nb}^{\,b} \rightarrow \vec{\omega}_{nb}^{\,b} = \vec{\omega}_{ib}^{\,b} - \vec{\omega}_{ie}^{\,b} - \vec{\omega}_{en}^{\,b}$$

- Start with angular velocity

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- Now

$$\dot{C}_b^n = C_b^n \Omega_{nb}^b = C_b^n \left(\Omega_{ib}^b - \Omega_{ie}^b - \Omega_{en}^b \right)$$
$$= C_b^n \Omega_{ib}^b - C_b^n \Omega_{ie}^b - C_b^n \Omega_{en}^b$$
$$= C_b^n \Omega_{ib}^b - \left(\Omega_{ie}^n + \Omega_{en}^n \right) C_b^n$$

- Start with angular velocity

$$\vec{\omega}^{\,b}_{ib} = \vec{\omega}^{\,b}_{ie} + \vec{\omega}^{\,b}_{en} + \vec{\omega}^{\,b}_{nb} \rightarrow \vec{\omega}^{\,b}_{nb} = \vec{\omega}^{\,b}_{ib} - \vec{\omega}^{\,b}_{ie} - \vec{\omega}^{\,b}_{en}$$

- Now

$$\dot{C}^n_b = C^n_b \Omega^b_{nb} = C^n_b \left(\Omega^b_{ib} - \Omega^b_{ie} - \Omega^b_{en} \right)$$

$$= C^n_b \Omega^b_{ib} - C^n_b \Omega^b_{ie} - C^n_b \Omega^b_{en}$$

$$= C^n_b \Omega^b_{ib} - \left(\Omega^n_{ie} + \Omega^n_{en} \right) C^n_b$$

Recall that:
$[(C\vec{\omega})\times] = C[\vec{\omega}\times]C^T$
$C[\vec{\omega}\times] = [(C\vec{\omega})\times]C$

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- Start with angular velocity

$$\vec{\omega}_{ib}^{\,b} = \vec{\omega}_{ie}^{\,b} + \vec{\omega}_{en}^{\,b} + \vec{\omega}_{nb}^{\,b} \rightarrow \vec{\omega}_{nb}^{\,b} = \vec{\omega}_{ib}^{\,b} - \vec{\omega}_{ie}^{\,b} - \vec{\omega}_{en}^{\,b}$$

- Now

$$\dot{C}_b^n = C_b^n \Omega_{nb}^b = C_b^n \left(\Omega_{ib}^b - \Omega_{ie}^b - \Omega_{en}^b \right)$$

$$= C_b^n \Omega_{ib}^b - C_b^n \Omega_{ie}^b - C_b^n \Omega_{en}^b$$

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Recall that:
$$[(C\vec{\omega})\times] = C[\vec{\omega}\times]C^T$$
$$C[\vec{\omega}\times] = [(C\vec{\omega})\times]C$$

$$\omega_{ie}^n = C_e^n \omega_{ie}^e = \omega_{ie} \begin{bmatrix} \cos L_b \\ 0 \\ -\sin L_b \end{bmatrix}$$

- $\Omega_{en}^n = [\vec{\omega}_{en}^n \times]$
- Courtesy of Prof. Bruder and Mathematica

$$\dot{C}_n^e = C_n^e \Omega_{en}^n \rightarrow \Omega_{en}^n = (C_n^e)^T \dot{C}_n^e = \begin{bmatrix} 0 & \dot{\lambda}_b \sin L_b & -\dot{L}_b \\ -\dot{\lambda}_b \sin L_b & 0 & -\dot{\lambda}_b \cos L_b \\ \dot{L}_b & \dot{\lambda}_b \cos L_b & 0 \end{bmatrix}$$

- $\Omega_{en}^n = [\vec{\omega}_{en}^n \times]$
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- therefore,

$$\omega_{en}^n = \begin{bmatrix} \dot{\lambda}_b \cos L_b \\ -\dot{L}_b \\ -\dot{\lambda}_b \sin L_b \end{bmatrix}$$

- $\Omega_{en}^n = [\vec{\omega}_{en}^n \times]$
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$$\dot{C}_n^e = C_n^e \Omega_{en}^n \rightarrow \Omega_{en}^n = (C_n^e)^T \dot{C}_n^e = \begin{bmatrix} 0 & \dot{\lambda}_b \sin L_b & -\dot{L}_b \\ -\dot{\lambda}_b \sin L_b & 0 & -\dot{\lambda}_b \cos L_b \\ \dot{L}_b & \dot{\lambda}_b \cos L_b & 0 \end{bmatrix}$$

- therefore,

$$\omega_{en}^n = \begin{bmatrix} \dot{\lambda}_b \cos L_b \\ -\dot{L}_b \\ -\dot{\lambda}_b \sin L_b \end{bmatrix}$$

- Finally since,

- then

$$\begin{bmatrix} \dot{L}_b \\ \dot{\lambda}_b \\ \dot{h}_b \end{bmatrix} = \begin{bmatrix} \frac{\vec{v}_{eb,N}^n}{R_N + h_b} \\ \frac{\vec{v}_{eb,E}^n}{\cos(L_b)(R_E + h_b)} \\ -\vec{v}_{eb,D}^n \end{bmatrix}$$

$$\omega_{en}^n = \begin{bmatrix} \frac{\vec{v}_{eb,E}^n}{R_E + h_b} \\ -\frac{\vec{v}_{eb,N}^n}{R_N + h_b} \\ -\frac{\tan(L_b)\vec{v}_{eb,E}^n}{R_E + h_b} \end{bmatrix} \quad (1)$$

$$C_b^n(+) - C_b^n(-) \approx \Delta t \, \dot{C}_b^n$$

$$C_b^n(+) \approx C_b^n(-) + \Delta t \left[C_b^n \Omega_{ib}^b - (\Omega_{ie}^n + \Omega_{en}^n) \, C_b^n(-) \right]$$

$$= C_b^n(-) \left(\mathcal{I} + \Omega_{ib}^b \Delta t \right) - (\Omega_{ie}^n + \Omega_{en}^n) \, C_b^n(-) \Delta t$$

- Body orientation frame at time "k" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$
- Start with the angular velocity

$$\omega_{nb}^b = \omega_{ib}^b - \omega_{ie}^b - \omega_{en}^b$$

$$\Omega_{nb}^b = \Omega_{ib}^b - \Omega_{ie}^b - \Omega_{en}^b$$

$$= \Omega_{ib}^b - C_n^b \Omega_{ie}^n C_b^n - C_n^b \Omega_{en}^n C_b^n$$

$$C_b^n(+) = C_b^n(-) e^{\Omega_{nb}^b \Delta t}$$

$$C_b^n(+) = C_b^n(-) \left[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1 - \cos(\Delta\theta)]\,\mathfrak{K}^2 \right]$$

$$\boxed{e^{\Omega_{nb}^b \Delta t} = e^{\mathfrak{K}\Delta\theta}}$$

- Body orientation frame at time "k" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$

$$\bar{q}\,_b^n(+) = \bar{q}\,_b^n(-) \otimes \Delta\bar{q}\,_b^b \boxed{\vec{\omega}\,_{nb}^b \Delta t = \vec{k}\Delta\theta}$$

$$\Delta\bar{q}\,_{b(k)}^{b(k-1)} = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix}$$

Need to periodically renormalize $\bar{q}$

$$\vec{\omega}\,^b_{nb}\Delta t = \vec{k}\Delta\theta \qquad \mathfrak{K} = [\vec{k}\times]$$

- High fidelity

$$C_b^n(+) = C_b^n(-)\left[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1 - \cos(\Delta\theta)]\,\mathfrak{K}^2\right] \qquad (2)$$

  or

$$\bar{q}\,^n_b(+) = \bar{q}\,^n_b(-) \otimes \Delta\bar{q}\,^{b(k-1)}_{b(k)}$$

$$\Delta\bar{q}\,^{b(k-1)}_{b(k)} = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix} \qquad (3)$$

- Low fidelity

$$C_b^n(+) \approx C_b^n(-)\left(\mathcal{I} + \Omega_{ib}^b\Delta t\right) - (\Omega_{ie}^n + \Omega_{en}^n)\,C_b^n(-)\Delta t \qquad (4)$$

2. Specific force transformation
   - Simply coordinatize the specific force

$$\vec{f}^{\,n}_{ib} = C^n_b(+)\vec{f}^{\,b}_{ib} \tag{5}$$

3. Velocity update
   - Note that: $\vec{v}^{\,n}_{eb} = C^n_e \vec{v}^{\,e}_{eb}$

- ③ Velocity update
  - Note that: $\vec{v}^{\,n}_{eb} = C^n_e \vec{v}^{\,e}_{eb}$

$$\dot{\vec{v}}^{\,n}_{eb} = \dot{C}^n_e \vec{v}^{\,e}_{eb} + C^n_e \, \dot{\vec{v}}^{\,e}_{eb}$$

$$\dot{\vec{v}}\,^e_{eb} = \vec{a}\,^e_{eb} = \vec{f}\,^e_{ib} + \vec{g}\,^e_b - 2\Omega^e_{ie}\vec{v}\,^e_{eb}$$

- **③** Velocity update
  - Note that: $\vec{v}\,^n_{eb} = C^n_e \vec{v}\,^e_{eb}$

$$\dot{\vec{v}}\,^n_{eb} = \dot{C}^n_e \vec{v}\,^e_{eb} + C^n_e\, \dot{\vec{v}}\,^e_{eb}$$

$$\dot{\vec{v}}^{\,e}_{eb} = \vec{a}^{\,e}_{eb} = \vec{f}^{\,e}_{ib} + \vec{g}^{\,e}_{b} - 2\Omega^{e}_{ie}\vec{v}^{\,e}_{eb}$$

- Velocity update
  - Note that: $\vec{v}^{\,n}_{eb} = C^{n}_{e}\vec{v}^{\,e}_{eb}$

$$\dot{\vec{v}}^{\,n}_{eb} = \dot{C}^{n}_{e}\vec{v}^{\,e}_{eb} + C^{n}_{e}\,\dot{\vec{v}}^{\,e}_{eb}$$

$$= \Omega^{n}_{ne}C^{n}_{e}\vec{v}^{\,e}_{eb} + C^{n}_{e}\left(\vec{f}^{\,e}_{ib} + \vec{g}^{\,e}_{b} - 2\Omega^{e}_{ie}\vec{v}^{\,e}_{eb}\right)$$

- ③ Velocity update
  - Note that: $\vec{v}^{\,n}_{eb} = C^n_e \vec{v}^{\,e}_{eb}$

$$\dot{\vec{v}}^{\,e}_{eb} = \vec{a}^{\,e}_{eb} = \vec{f}^{\,e}_{ib} + \vec{g}^{\,e}_b - 2\Omega^e_{ie}\vec{v}^{\,e}_{eb}$$

$$\dot{\vec{v}}^{\,n}_{eb} = \dot{C}^n_e \vec{v}^{\,e}_{eb} + C^n_e \dot{\vec{v}}^{\,e}_{eb}$$

$$= \Omega^n_{ne} C^n_e \vec{v}^{\,e}_{eb} + C^n_e \left(\vec{f}^{\,e}_{ib} + \vec{g}^{\,e}_b - 2\Omega^e_{ie}\vec{v}^{\,e}_{eb} \right)$$

$$= \vec{f}^{\,n}_{ib} + \vec{g}^{\,n}_b - \Omega^n_{en}\vec{v}^{\,n}_{eb} - 2C^n_e\Omega^e_{ie}\vec{v}^{\,e}_{eb}$$

- 3. Velocity update
  - Note that: $\vec{v}_{eb}^{\,n} = C_e^n \vec{v}_{eb}^{\,e}$

$$\dot{\vec{v}}_{eb}^{\,e} = \vec{a}_{eb}^{\,e} = \vec{f}_{ib}^{\,e} + \vec{g}_{b}^{\,e} - 2\Omega_{ie}^e \vec{v}_{eb}^{\,e}$$

$$\dot{\vec{v}}_{eb}^{\,n} = \dot{C}_e^n \vec{v}_{eb}^{\,e} + C_e^n \, \dot{\vec{v}}_{eb}^{\,e}$$

$$= \Omega_{ne}^n C_e^n \vec{v}_{eb}^{\,e} + C_e^n \left(\vec{f}_{ib}^{\,e} + \vec{g}_{b}^{\,e} - 2\Omega_{ie}^e \vec{v}_{eb}^{\,e} \right)$$

$$= \vec{f}_{ib}^{\,n} + \vec{g}_{b}^{\,n} - \Omega_{en}^n \vec{v}_{eb}^{\,n} - 2C_e^n \Omega_{ie}^e \vec{v}_{eb}^{\,e}$$

$$= \vec{f}_{ib}^{\,n} + \vec{g}_{b}^{\,n} - \Omega_{en}^n \vec{v}_{eb}^{\,n} - 2\Omega_{ie}^n C_e^n \vec{v}_{eb}^{\,e}$$

- **Velocity update**
  - Note that: $\vec{v}_{eb}^{\,n} = C_e^n \vec{v}_{eb}^{\,e}$

$$\dot{\vec{v}}_{eb}^{\,e} = \vec{a}_{eb}^{\,e} = \vec{f}_{ib}^{\,e} + \vec{g}_b^{\,e} - 2\Omega_{ie}^e \vec{v}_{eb}^{\,e}$$

$$\dot{\vec{v}}_{eb}^{\,n} = \dot{C}_e^n \vec{v}_{eb}^{\,e} + C_e^n \dot{\vec{v}}_{eb}^{\,e}$$

$$= \Omega_{ne}^n C_e^n \vec{v}_{eb}^{\,e} + C_e^n \left(\vec{f}_{ib}^{\,e} + \vec{g}_b^{\,e} - 2\Omega_{ie}^e \vec{v}_{eb}^{\,e} \right)$$

$$= \vec{f}_{ib}^{\,n} + \vec{g}_b^{\,n} - \Omega_{en}^n \vec{v}_{eb}^{\,n} - 2C_e^n \Omega_{ie}^e \vec{v}_{eb}^{\,e}$$

$$= \vec{f}_{ib}^{\,n} + \vec{g}_b^{\,n} - \Omega_{en}^n \vec{v}_{eb}^{\,n} - 2\Omega_{ie}^n C_e^n \vec{v}_{eb}^{\,e}$$

$$= \vec{f}_{ib}^{\,n} + \vec{g}_b^{\,n} - \left(\Omega_{en}^n + 2\Omega_{ie}^n \right) \vec{v}_{eb}^{\,n}$$

- ③ Velocity update
  - Note that: $\vec{v}^{\,n}_{eb} = C^n_e \vec{v}^{\,e}_{eb}$

$$\dot{\vec{v}}^{\,e}_{eb} = \vec{a}^{\,e}_{eb} = \vec{f}^{\,e}_{ib} + \vec{g}^{\,e}_{b} - 2\Omega^e_{ie}\vec{v}^{\,e}_{eb}$$

$$\dot{\vec{v}}^{\,n}_{eb} = \dot{C}^n_e \vec{v}^{\,e}_{eb} + C^n_e \dot{\vec{v}}^{\,e}_{eb}$$

$$= \Omega^n_{ne} C^n_e \vec{v}^{\,e}_{eb} + C^n_e \left(\vec{f}^{\,e}_{ib} + \vec{g}^{\,e}_{b} - 2\Omega^e_{ie}\vec{v}^{\,e}_{eb} \right)$$

$$= \vec{f}^{\,n}_{ib} + \vec{g}^{\,n}_{b} - \Omega^n_{en}\vec{v}^{\,n}_{eb} - 2C^n_e\Omega^e_{ie}\vec{v}^{\,e}_{eb}$$

$$= \vec{f}^{\,n}_{ib} + \vec{g}^{\,n}_{b} - \Omega^n_{en}\vec{v}^{\,n}_{eb} - 2\Omega^n_{ie}C^n_e\vec{v}^{\,e}_{eb}$$

$$= \vec{f}^{\,n}_{ib} + \vec{g}^{\,n}_{b} - \left(\Omega^n_{en} + 2\Omega^n_{ie} \right) \vec{v}^{\,n}_{eb}$$

- Finally,

$$\vec{v}^{\,n}_{eb}(+) = \vec{v}^{\,n}_{eb}(-) + \Delta t \left[\vec{f}^{\,n}_{ib} + \vec{g}^{\,n}_{b} - \left(\Omega^n_{en} + 2\Omega^n_{ie} \right) \vec{v}^{\,n}_{eb}(-) \right]$$

4. Position update
   - Recalling that

$$
\begin{bmatrix} \dot{L}_b \\ \dot{\lambda}_b \\ \dot{h}_b \end{bmatrix} = \begin{bmatrix} \dfrac{\vec{v}^{\,n}_{eb,N}}{R_N + h_b} \\[2ex] \dfrac{\vec{v}^{\,n}_{eb,E}}{\cos(L_b)(R_E + h_b)} \\[2ex] -\vec{v}^{\,n}_{eb,D} \end{bmatrix}
$$

5. then

$$
h_b(+) = h_b(-) - \Delta t \left[\vec{v}^{\,n}_{eb,D} \right]
$$

$$
L_b(+) = L_b(-) + \Delta t \left[\frac{\vec{v}^{\,n}_{eb,N}}{R_N + h_b} \right]
$$

$$
\lambda_b(+) = \lambda_b(-) + \Delta t \left[\frac{\vec{v}^{\,n}_{eb,E}}{\cos(L_b)(R_E + h_b)} \right]
$$

$$C_b^n(+) = C_b^n(-) \left[\mathcal{I} + \sin(\Delta\theta)\hat{\mathfrak{K}} + [1 - \cos(\Delta\theta)] \hat{\mathfrak{K}}^2 \right] \quad (6)$$

or

$$\bar{q}\,_b^n(+) = \bar{q}\,_b^n(-) \otimes \Delta\bar{q}\,_{b(k)}^{b(k-1)}, \quad \Delta\bar{q}\,_{b(k)}^{b(k-1)} = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix} \quad (7)$$

or

$$C_b^n(+) \approx C_b^n(-) \left(\mathcal{I} + \Omega_{ib}^b \Delta t \right) - (\Omega_{ie}^n + \Omega_{en}^n) \, C_b^n(-)\Delta t \quad (8)$$
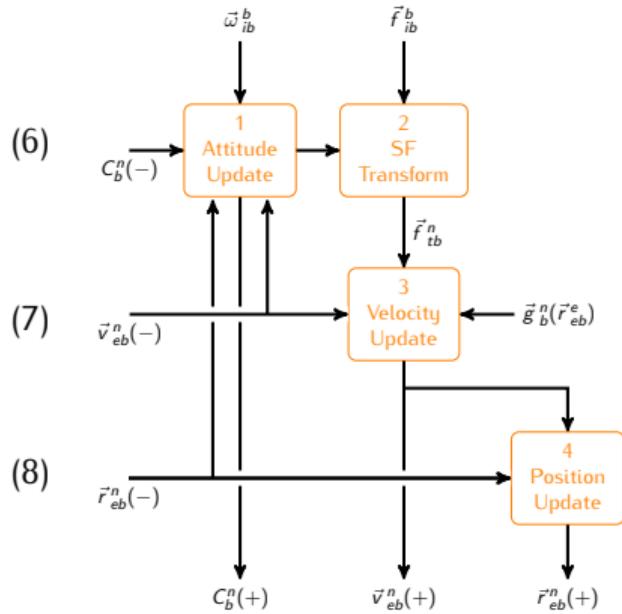
$$\vec{f}_{ib}^{\,n} = C_b^n(+)\vec{f}_{ib}^{\,b} \tag{9}$$

$$\vec{a}_{eb}^{\,n} = \vec{f}_{ib}^{\,n} + \vec{g}_b^{\,n} - (\Omega_{en}^n + 2\Omega_{ie}^n)\vec{v}_{eb}^{\,n} \tag{10}$$

$$\vec{v}_{eb}^{\,n}(+) = \vec{v}_{eb}^{\,n}(-) + \Delta t \left[\vec{f}_{ib}^{\,n} + \vec{g}_b^{\,n} - (\Omega_{en}^n + 2\Omega_{ie}^n)\,\vec{v}_{eb}^{\,n}(-) \right] \tag{11}$$

$$L_b(+) = L_b(-) + \Delta t \left[\frac{\vec{v}_{eb,N}^{\,n}}{R_N + h_b} \right] \tag{12}$$

$$\lambda_b(+) = \lambda_b(-) + \Delta t \left[\frac{\vec{v}_{eb,E}^{\,n}}{\cos(L_b)(R_E + h_b)} \right] \tag{13}$$

$$h_b(+) = h_b(-) - \Delta t \left[\vec{v}_{eb,D}^{\,n} \right] \tag{14}$$
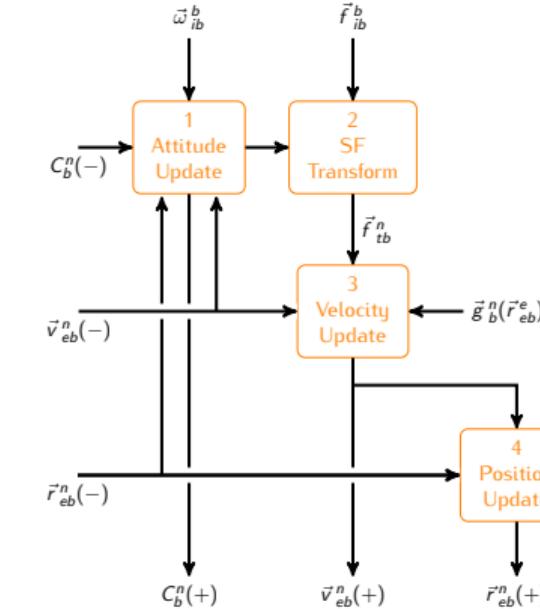
- In continuous time notation
  - Attitude: $\dot{C}_b^n = C_b^n \Omega_{ib}^b - (\Omega_{ie}^n + \Omega_{en}^n) C_b^n$
  - Velocity: $\dot{\vec{v}}_{eb}^n = \vec{f}_{ib}^n + \vec{g}_b^n - (\Omega_{en}^n + 2\Omega_{ie}^n) \vec{v}_{eb}^n$
  - Position:

$$
\begin{bmatrix} \dot{L}_b \\\\ \dot{\lambda}_b \\\\ \dot{h}_b \end{bmatrix} = \begin{bmatrix} \dfrac{\vec{v}_{eb,N}^n}{R_N + h_b} \\\\ \dfrac{\vec{v}_{eb,E}^n}{\cos(L_b)(R_E + h_b)} \\\\ -\vec{v}_{eb,D}^n \end{bmatrix}
$$

- In State–space notation

$$
\begin{bmatrix}
\dot{L}_b \\[2mm]
\dot{\lambda}_b \\[2mm]
\dot{h}_b \\[2mm]
\dot{\vec{v}}^{\,n}_{eb} \\[2mm]
\dot{C}^n_b
\end{bmatrix}
=
\begin{bmatrix}
\dfrac{\vec{v}^{\,n}_{eb,N}}{R_N + h_b} \\[3mm]
\dfrac{\vec{v}^{\,n}_{eb,E}}{\cos(L_b)(R_E + h_b)} \\[3mm]
-\vec{v}^{\,n}_{eb,D} \\[3mm]
\vec{f}^{\,n}_{ib} + \vec{g}^{\,n}_b - (\Omega^n_{en} + 2\Omega^n_{ie})\vec{v}^{\,n}_{eb} \\[3mm]
C^n_b \Omega^b_{ib} - (\Omega^n_{ie} + \Omega^n_{en})C^n_b
\end{bmatrix}
\tag{15}
$$

$$[\bar{q}\otimes] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & -q_z & q_y \\ q_y & q_z & q_s & -q_x \\ q_z & -q_y & q_x & q_s \end{bmatrix}$$

$$[\bar{q}\circledast] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & q_z & -q_y \\ q_y & -q_z & q_s & q_x \\ q_z & q_y & -q_x & q_s \end{bmatrix}$$