EE 565: Position, Navigation and Timing

Navigation Equations: Tangential Mechanization

Aly El-Osery Kevin Wedeward

Electrical Engineering Department, New Mexico Tech
Socorro, New Mexico, USA

In Collaboration with
Stephen Bruder
Electrical and Computer Engineering Department
Embry-Riddle Aeronautical Univesity
Prescott, Arizona, USA

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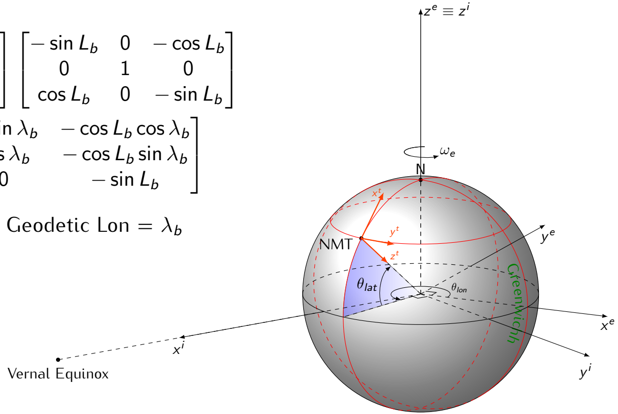
- Determine the position, velocity and attitude of the **body** frame *wrt* the **tangential** frame.
  - **Position** — Vector from the origin of the tangential frame to the origin of the body frame resolved in the tangential frame: $\vec{r}^{\,t}_{tb}$
  - **Velocity** — Velocity of the body frame *wrt* the tangential frame resolved in the tangential frame: $\vec{v}^{\,t}_{tb}$
  - **Attitude** — Orientation of the body frame *wrt* the tangential frame: C^t_b

- Description of the tangential frame
  - Orientation of the t-frame *wrt* the e-frame

$$
\begin{aligned}
C_t^e &= R_{(\vec{z},\lambda_b)} R_{(\vec{y},-L_b-90^\circ)} \\
&= \begin{bmatrix} \cos\lambda_b & -\sin\lambda_b & 0 \\ \sin\lambda_b & \cos\lambda_b & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin L_b & 0 & -\cos L_b \\ 0 & 1 & 0 \\ \cos L_b & 0 & -\sin L_b \end{bmatrix} \\
&= \begin{bmatrix} -\sin L_b \cos\lambda_b & -\sin\lambda_b & -\cos L_b \cos\lambda_b \\ -\sin L_b \sin\lambda_b & \cos\lambda_b & -\cos L_b \sin\lambda_b \\ \cos L_b & 0 & -\sin L_b \end{bmatrix}
\end{aligned}
$$

where geodetic Lat $= L_b$ and Geodetic Lon $= \lambda_b$



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- Body orientation frame at time "k" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$
- Start with angular velocity

$$\vec{\omega}\,^t_{ib} = \vec{\omega}\,^t_{ie} + \vec{\omega}\,^t_{et} + \vec{\omega}\,^t_{tb}$$

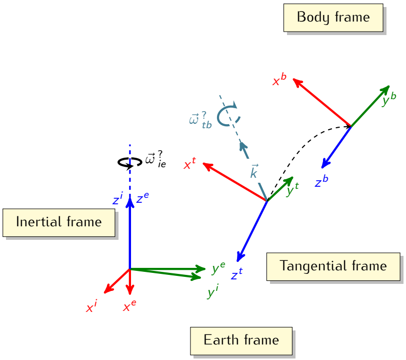
$$\vec{\omega}\,^t_{tb} = C^t_b \vec{\omega}\,^b_{ib} - C^t_e \vec{\omega}\,^e_{ie}$$

$$\Omega^t_{tb} = C^t_b \Omega^b_{ib} C^b_t - C^t_e \Omega^e_{ie} C^e_t$$

$$C^t_b(+) - C^t_b(-) \approx \Delta t \Omega^t_{tb} C^t_b(-)$$

$$C^t_b(+) \approx C^t_b(-) + \Delta t \left(C^t_b \Omega^b_{ib} C^b_t - C^t_e \Omega^e_{ie} C^e_t \right) C^t_b(-)$$

$$= C^t_b(-) \left(\mathcal{I} + \Omega^b_{ib} \Delta t \right) - \Omega^t_{ie} C^t_b(-) \Delta t$$



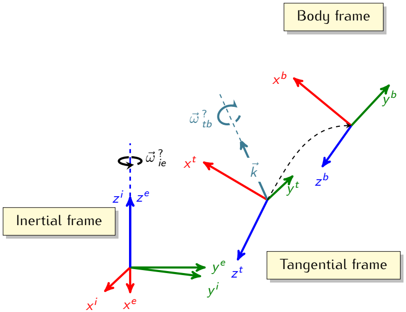
- Body orientation frame at time “k” *wrt* time “$k-1$”
  - $\Delta t = t_k - t_{k-1}$
- Start with the angular velocity

$$\Omega^t_{tb} = C^t_b \Omega^b_{ib} C^b_t - \Omega^t_{ie}$$

$$C^t_b(+) = C^t_b(-) e^{\Omega^b_{tb}\Delta t} = e^{\Omega^t_{tb}\Delta t} C^t_b(-)$$

$$C^t_b(+) = \left[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1-\cos(\Delta\theta)]\,\mathfrak{K}^2\right] C^t_b(-)$$

$$\boxed{e^{\Omega^t_{tb}\Delta t} = e^{\mathfrak{K}\theta}}$$



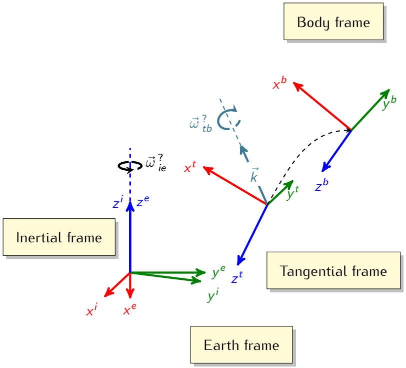
- Body orientation frame at time "k" *wrt* time "$k-1$"
  - $\Delta t = t_k - t_{k-1}$

$$\vec{\omega}^{\,t}_{tb}\Delta t = \vec{k}\Delta\theta$$

$$\bar{q}^{\,t}_{b}(+) = \Delta\bar{q}^{\,t}_{b} \otimes \bar{q}^{\,t}_{b}(-)$$

$$\Delta\bar{q}^{\,t}_{b} = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix}$$

Need to periodically renormalize $\bar{q}$



$$\vec{\omega}\,^t_{tb}\Delta t = \vec{k}\Delta\theta \qquad \mathfrak{K} = [\vec{k}\times]$$

- High fidelity

$$C^t_b(+) = \left[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1 - \cos(\Delta\theta)]\,\mathfrak{K}^2\right] C^t_b(-) \tag{1}$$

or

$$\bar{q}\,^t_b(+) = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix} \otimes \bar{q}\,^t_b(-) \tag{2}$$

- Low fidelity

$$C^t_b(+) \approx C^t_b(-)\left(\mathcal{I} + \Omega^b_{ib}\Delta t\right) - \Omega^t_{ie}C^t_b(-)\Delta t \tag{3}$$

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2. Specific force transformation
   - Simply coordinatize the specific force

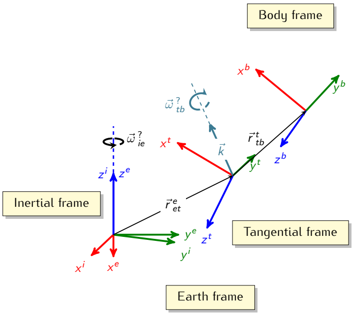
$$\vec{f}^{\,t}_{ib} = C^t_b(+)\vec{f}^{\,b}_{ib} \tag{4}$$

3. Velocity update

$$\vec{r}^{\,i}_{ib} = \underbrace{\vec{r}^{\,i}_{ie}}_{0} + C^i_e\vec{r}^{\,e}_{et} + C^i_t\vec{r}^{\,t}_{tb}$$

$$\Rightarrow \vec{r}^{\,t}_{tb} = C^t_i\vec{r}^{\,i}_{ib} - C^t_e\vec{r}^{\,e}_{et}$$

$$\begin{aligned}\vec{v}^{\,t}_{tb} &= \dot{\vec{r}}^{\,t}_{tb}\\ &= \dot{C}^t_i\vec{r}^{\,i}_{ib} + C^t_i\dot{\vec{r}}^{\,i}_{ib}\\ &= \Omega^t_{ei}C^t_i\vec{r}^{\,i}_{ib} + C^t_i\vec{v}^{\,i}_{ib}\\ &= -\Omega^t_{ie}(\vec{r}^{\,t}_{et} + \vec{r}^{\,t}_{tb}) + C^t_i\vec{v}^{\,i}_{ib}\end{aligned}$$



$$\vec{a}^{\,t}_{tb} = \dot{\vec{v}}^{\,t}_{tb} = \frac{d}{dt}\left(-\Omega^t_{ie}(\vec{r}^{\,t}_{et} + \vec{r}^{\,t}_{tb}) + C^t_i \vec{v}^{\,i}_{ib}\right)$$
$$= -\Omega^t_{ie}\dot{\vec{r}}^{\,t}_{tb} + \dot{C}^t_i \vec{v}^{\,i}_{ib} + C^t_i \dot{\vec{v}}^{\,i}_{ib}$$
$$= -\Omega^t_{ie}\vec{v}^{\,t}_{tb} + \Omega^t_{ti} C^t_i \vec{v}^{\,i}_{ib} + C^t_i \vec{a}^{\,i}_{ib}$$
$$= -\Omega^t_{ie}\vec{v}^{\,t}_{tb} - \Omega^t_{it}\left[\vec{v}^{\,t}_{tb} + \Omega^t_{ie}(\vec{r}^{\,t}_{et} + \vec{r}^{\,t}_{tb})\right] + C^t_i \vec{a}^{\,i}_{ib}$$
$$= -2\Omega^t_{ie}\vec{v}^{\,t}_{tb} - \Omega^t_{ie}\Omega^t_{ie}\vec{r}^{\,t}_{eb} + \vec{a}^{\,t}_{ib}$$
$$= -2\Omega^t_{ie}\vec{v}^{\,t}_{tb} + \vec{f}^{\,t}_{ib} + \vec{g}^{\,t}_{b}$$

$$\vec{v}^{\,t}_{tb}(+) = \vec{v}^{\,t}_{tb}(-) + \vec{a}^{\,t}_{tb}\Delta t$$
$$= \vec{v}^{\,t}_{tb}(-) + \left[\vec{f}^{\,t}_{ib} + \vec{g}^{\,t}_{b} - 2\Omega^t_{ie}\vec{v}^{\,t}_{tb}(-)\right]\Delta t \qquad (5)$$

$$\dot{\vec{r}}^{\,t}_{et} = 0$$

$$\vec{\omega}^{\,t}_{it} = \vec{\omega}^{\,t}_{ie}$$

$$C^t_i \vec{v}^{\,i}_{ib} = \Omega^t_{ie}(\vec{r}^{\,t}_{et} + \vec{r}^{\,t}_{tb}) + \vec{v}^{\,t}_{tb}$$

$$\vec{f}^{\,?}_{ib} = \vec{a}^{\,?}_{ib} - \vec{\gamma}^{\,?}_{ib}$$

$$\vec{a}^{\,t}_{ib} = \vec{f}^{\,t}_{ib} + \vec{\gamma}^{\,t}_{ib}$$

$$\vec{g}^{\,t}_{b} = \vec{\gamma}^{\,t}_{ib} - \Omega^t_{ie}\Omega^t_{ie}\vec{r}^{\,t}_{eb}$$

$$\vec{a}^{\,t}_{ib} = \vec{f}^{\,t}_{ib} + \vec{g}^{\,t}_{b} + \Omega^t_{ie}\Omega^t_{ie}\vec{r}^{\,t}_{eb}$$

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4. Position update
   - by simple numerical integration

$$\vec{r}^{\,t}_{tb}(+) = \vec{r}^{\,t}_{tb}(-) + \vec{v}^{\,t}_{tb}(-)\Delta t + \vec{a}^{\,t}_{tb}\frac{\Delta t^2}{2} \tag{6}$$

$$C_b^t(+) = \left[\mathcal{I} + \sin(\Delta\theta)\mathfrak{K} + [1-\cos(\Delta\theta)]\,\mathfrak{K}^2\right] C_b^t(-)$$

or

$$\bar{q}\,{}^t_b(+) = \begin{bmatrix} \cos(\frac{\Delta\theta}{2}) \\ \vec{k}\sin(\frac{\Delta\theta}{2}) \end{bmatrix} \otimes \bar{q}\,{}^t_b(-)$$

or

$$C_b^t(+) \approx C_b^t(-)\left(\mathcal{I} + \Omega^b_{ib}\Delta t\right) - \Omega^t_{ie}C_b^t(-)\Delta t$$

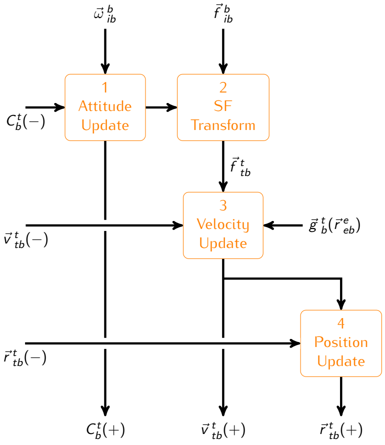
and

$$\vec{f}\,{}^t_{ib} = C_b^t(+)\vec{f}\,{}^b_{ib}$$

$$\vec{a}\,{}^t_{tb} = -2\Omega^t_{ie}\vec{v}\,{}^t_{tb} + \vec{f}\,{}^t_{ib} + \vec{g}\,{}^t_b$$

$$\vec{v}\,{}^t_{tb}(+) = \vec{v}\,{}^t_{tb}(-) + \left[\vec{f}\,{}^t_{ib} + \vec{g}\,{}^t_b - 2\Omega^t_{ie}\vec{v}\,{}^t_{tb}(-)\right]\Delta t$$

$$\vec{r}\,{}^t_{tb}(+) = \vec{r}\,{}^t_{tb}(-) + \vec{v}\,{}^t_{tb}(-)\Delta t + \vec{a}\,{}^t_{tb}\frac{\Delta t^2}{2}$$



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- In continuous time notation
  - Attitude: $\dot{C}_b^t = C_b^t \Omega_{tb}^b$ or $\dot{\bar{q}}_b^t = \frac{1}{2}[\breve{\omega}_{tb}^b \circledast]\bar{q}\,_b^t(t)$
  - Velocity: $\dot{\vec{v}}\,_{tb}^t = C_b^t \vec{f}\,_{ib}^b + \vec{g}\,_b^t - 2\Omega_{ie}^t \vec{v}\,_{tb}^t$
  - Position: $\dot{\vec{r}}\,_{ib}^t = \vec{v}\,_{tb}^t$

$$\vec{\omega}\,_{ib}^b = \vec{\omega}\,_{ie}^b + \vec{\omega}\,_{tb}^b$$

$$\Omega_{tb}^b = \Omega_{ib}^b - \Omega_{ie}^b$$

- In State-space notation

$$\begin{bmatrix} \dot{\vec{r}}\,_{tb}^t \\ \dot{\vec{v}}\,_{tb}^t \\ \dot{C}_b^t \end{bmatrix} = \begin{bmatrix} \vec{v}\,_{tb}^t \\ C_b^t \vec{f}\,_{ib}^b + \vec{g}\,_b^t - 2\Omega_{ie}^t \vec{v}\,_{tb}^t \\ C_b^t \Omega_{tb}^b \end{bmatrix} \tag{7}$$

or

$$\begin{bmatrix} \dot{\vec{r}}\,_{tb}^t \\ \dot{\vec{v}}\,_{tb}^t \\ \dot{\bar{q}}_b^t \end{bmatrix} = \begin{bmatrix} \vec{v}\,_{tb}^t \\ C_b^t \vec{f}\,_{ib}^b + \vec{g}\,_b^t - 2\Omega_{ie}^t \vec{v}\,_{tb}^t \\ \frac{1}{2}[\breve{\omega}_{tb}^b \circledast]\bar{q}\,_b^t(t) \end{bmatrix} \tag{8}$$

$$[\bar{q}\otimes] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & -q_z & q_y \\ q_y & q_z & q_s & -q_x \\ q_z & -q_y & q_x & q_s \end{bmatrix}$$

$$[\bar{q}\circledast] = \begin{bmatrix} q_s & -q_x & -q_y & -q_z \\ q_x & q_s & q_z & -q_y \\ q_y & -q_z & q_s & q_x \\ q_z & q_y & -q_x & q_s \end{bmatrix}$$