Lecture
Error Mechanization (Tangential)
EE 565: Position, Navigation and Timing

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Tangential Mechanization — Continuous Case

- In continuous time notation
  - Attitude: $\dot{C}_b^t = C_b^t \Omega_{tb}^b$
  - Velocity: $\dot{\vec{v}}_{tb}^t = C_b^t \vec{f}_{ib}^b + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^t$
  - Position: $\dot{\vec{r}}_{ib}^t = \vec{v}_{tb}^t$
- In State–space notation

$$\begin{bmatrix} \dot{\vec{r}}_{tb}^t \\ \dot{\vec{v}}_{tb}^t \\ \dot{C}_b^t \end{bmatrix} = \begin{bmatrix} \vec{v}_{tb}^t \\ C_b^t \vec{f}_{ib}^b + \vec{g}_b^t - 2\Omega_{ie}^t \vec{v}_{tb}^t \\ C_b^t \Omega_{tb}^b \end{bmatrix} \tag{1}$$

where $\vec{\omega}_{ib}^b = \vec{\omega}_{ie}^b + \vec{\omega}_{tb}^b$, and $\Omega_{tb}^b = \Omega_{ib}^b - \Omega_{ie}^b$.

Question
What is the effect of sensor noise in the measurements $\vec{\omega}_{ib}^b$ and $\vec{f}_{ib}^b$ on the navigation solution position, velocity and attitude?

1 Attitude

Tangential Attitude Error

$$\dot{C}_b^t = C_b^t \Omega_{tb}^b = C_b^t(\Omega_{ib}^b - \Omega_{ie}^b) = \frac{d}{dt}\left[(\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\hat{C}_b^t\right] =$$

$$(\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\hat{C}_b^t \Omega_{tb}^b = [\delta\dot{\vec{\psi}}_{tb}^t \times]\hat{C}_b^t + (\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\dot{\hat{C}}_b^t =$$
$$(\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\hat{C}_b^t(\hat{\Omega}_{tb}^b + \delta\Omega_{ib}^b - \delta\Omega_{ie}^b) =$$
$$(\mathcal{I} + [\delta\vec{\psi}_{tb}^t \times])\hat{C}_b^t\hat{\Omega}_{tb}^b + \hat{C}_b^t(\delta\Omega_{ib}^b - \delta\Omega_{ie}^b) =$$

$$[\delta \dot{\vec{\psi}}^t_{tb} \times] = \hat{C}^t_b(\delta \Omega^b_{ib} - \delta \Omega^b_{ie})\hat{C}^b_t = [\hat{C}^t_b(\delta \vec{\omega}^{\,b}_{ib} - \delta \vec{\omega}^{\,b}_{ie}) \times] \tag{2}$$

$$\delta \dot{\vec{\psi}}^t_{tb} = \hat{C}^t_b(\delta \vec{\omega}^b_{ib} - \delta \vec{\omega}^b_{ie}) \tag{3}$$

Tangential Attitude Error (cont.)

$$\begin{aligned}
\delta \dot{\vec{\psi}}^t_{tb} &= \hat{C}^t_b(\delta \vec{\omega}^b_{ib} - \delta \vec{\omega}^b_{ie}) \\
&= \hat{C}^t_b \delta \vec{\omega}^b_{ib} - \hat{C}^t_b(\vec{\omega}^b_{ie} - \hat{\vec{\omega}}^b_{ie}) \\
&= \hat{C}^t_b \delta \vec{\omega}^b_{ib} - (\hat{C}^t_b C^b_t - \mathcal{I})\vec{\omega}^t_{ie} \\
&= \hat{C}^t_b \delta \vec{\omega}^b_{ib} + \delta \vec{\psi}^t_{tb} \times \vec{\omega}^t_{ie}
\end{aligned}$$

$$\delta \dot{\vec{\psi}}^t_{tb} = \hat{C}^t_b \delta \vec{\omega}^b_{ib} - \vec{\Omega}^t_{ie} \delta \vec{\psi}^t_{tb} \tag{4}$$

2 Velocity

Velocity

$$\dot{\vec{v}}^t_{tb} = C^t_b \vec{f}^b_{ib} + \vec{g}^t_b - 2\Omega^t_{ie} \vec{v}^t_{tb} \tag{5}$$

$$\begin{aligned}
\dot{\hat{\vec{v}}}^t_{tb} &= \hat{C}^t_b \hat{\vec{f}}^b_{ib} + \hat{\vec{g}}^t_b - 2\Omega^t_{ie} \hat{\vec{v}}^t_{tb} \\
&= (\mathcal{I} - [\delta \vec{\psi}^t_{tb} \times])C^t_b(\vec{f}^b_{ib} - \delta \vec{f}^b_{ib}) + \hat{\vec{g}}^t_b - 2\Omega^t_{ie} \hat{\vec{v}}^t_{tb}
\end{aligned} \tag{6}$$

$$\begin{aligned}
\delta \dot{\vec{v}}^t_{tb} = \dot{\vec{v}}^t_{tb} - \dot{\hat{\vec{v}}}^t_{tb} &= [\delta \vec{\psi}^t_{tb} \times]C^t_b \vec{f}^b_{ib} + \hat{C}^t_b \delta \vec{f}^b_{ib} + \delta \vec{g}^t_b - 2\Omega^t_{ie} \delta \vec{v}^t_{tb} \\
&= [\delta \vec{\psi}^t_{tb} \times]\hat{C}^t_b \hat{\vec{f}}^b_{ib} + \hat{C}^t_b \delta \vec{f}^b_{ib} + \delta \vec{g}^t_b - 2\Omega^t_{ie} \delta \vec{v}^t_{tb}
\end{aligned}$$

$$\delta \dot{\vec{v}}^t_{tb} = -[\hat{C}^t_b \hat{\vec{f}}^b_{ib} \times]\delta \vec{\psi}^t_{tb} + \hat{C}^t_b \delta \vec{f}^b_{ib} + \delta \vec{g}^t_b - 2\Omega^t_{ie} \hat{\vec{v}}^t_{tb} \tag{7}$$

3 Gravity

Gravity Error

$$\delta \vec{g}^t_b \approx \hat{C}^t_e \frac{2 g_0(\hat{L}_b)}{r^e_{eS}(\hat{L}_b)} \frac{\hat{\vec{r}}^e_{eb}}{|\hat{\vec{r}}^e_{eb}|^2} (\hat{\vec{r}}^e_{eb})^T \hat{C}^e_t \delta \vec{r}^t_{tb} \tag{8}$$

4 Position

Position

$$\dot{\vec{r}}^t_{tb} = \vec{v}^t_{tb} \tag{9}$$

$$\delta \dot{\vec{r}}^t_{tb} = \delta \vec{v}^t_{tb} \tag{10}$$

5 Summary

Summary - in terms of $\delta\vec{f}^{\,b}_{ib}, \delta\vec{\omega}^{\,b}_{ib}$

$$
\begin{pmatrix} \delta\dot{\vec{\psi}}^{\,t}_{tb} \\ \delta\dot{\vec{v}}^{\,t}_{tb} \\ \delta\dot{\vec{r}}^{\,t}_{tb} \end{pmatrix} = \begin{bmatrix} -\Omega^t_{ie} & 0_{3\times3} & 0_{3\times3} \\ -[\hat{C}^t_b\hat{\vec{f}}^{\,b}_{ib}\times] & -2\Omega^t_{ie} & \hat{C}^t_e\frac{2g_0(\hat{L}_b)}{r^e_{eS}(\hat{L}_b)}\frac{\hat{\vec{r}}^e_{eb}}{|\hat{\vec{r}}^e_{eb}|^2}(\hat{\vec{r}}^e_{eb})^T\hat{C}^e_t \\ 0_{3\times3} & \mathcal{I}_{3\times3} & 0_{3\times3} \end{bmatrix} \begin{pmatrix} \delta\vec{\psi}^{\,t}_{tb} \\ \delta\vec{v}^{\,t}_{tb} \\ \delta\vec{r}^{\,t}_{tb} \end{pmatrix} +
$$

$$
\begin{bmatrix} 0 & \hat{C}^t_b \\ \hat{C}^t_b & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \delta\vec{f}^{\,b}_{ib} \\ \delta\vec{\omega}^{\,b}_{ib} \end{pmatrix}
$$

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A Basic Definitions

Notation Used

- *Truth value*

$$\vec{x}$$

- *Measured value*

$$\tilde{\vec{x}}$$

- *Estimated or computed value*

$$\hat{\vec{x}}$$

- *Error*

$$\delta\vec{x} = \vec{x} - \hat{\vec{x}}$$

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B Linearization

Linearization using Taylor Series Expansion

Given a non-linear system $\dot{\vec{x}} = f(\vec{x}, t)$

Let's assume we have an estimate of $\vec{x}$, i.e., $\hat{\vec{x}}$ such that $\vec{x} = \hat{\vec{x}} + \delta\vec{x}$

$$\dot{\vec{x}} = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}} + \delta\vec{x}, t) \tag{12}$$

Using Taylor series expansion

$$f(\hat{\vec{x}} + \delta\vec{x}, t) = \dot{\hat{\vec{x}}} + \delta\dot{\vec{x}} = f(\hat{\vec{x}}, t) + \left.\frac{\partial f(\vec{x}, t)}{\partial\vec{x}}\right|_{\vec{x}=\hat{\vec{x}}}\delta\vec{x} + H.O.T$$

$$\approx \dot{\hat{\vec{x}}} + \left.\frac{\partial f(\vec{x}, t)}{\partial\vec{x}}\right|_{\vec{x}=\hat{\vec{x}}}\delta\vec{x}$$

$$\Rightarrow \delta\dot{\vec{x}} \approx \left.\frac{\partial f(\vec{x}, t)}{\partial\vec{x}}\right|_{\vec{x}=\hat{\vec{x}}}\delta\vec{x} \tag{13}$$

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C   Inertial Measurements

Actual Measurements

Initially the accelerometer and gyroscope measurements, $\tilde{\vec{f}}_{ib}^{\,b}$ and $\tilde{\vec{\omega}}_{ib}^{\,b}$, respectively, will be modeled as

$$\tilde{\vec{f}}_{ib}^{\,b} = \vec{f}_{ib}^{\,b} + \Delta \vec{f}_{ib}^{\,b} \tag{14}$$

$$\tilde{\vec{\omega}}_{ib}^{\,b} = \vec{\omega}_{ib}^{\,b} + \Delta \vec{\omega}_{ib}^{\,b} \tag{15}$$

where $\vec{f}_{ib}^{\,b}$ and $\vec{\omega}_{ib}^{\,b}$ are the specific force and angular rates, respectively; and $\Delta \vec{f}_{ib}^{\,b}$ and $\Delta \vec{\omega}_{ib}^{\,b}$ represents the errors. In later lectures we will discuss more detailed description of these errors.

Error Modeling Example

Accelerometers

$$\tilde{\vec{f}}_{ib}^{\,b} = \vec{b}_a + (\mathcal{I} + M_a)\vec{f}_{ib}^{\,b} + \vec{nl}_a + \vec{w}_a \tag{16}$$

Gyroscopes

$$\tilde{\vec{\omega}}_{ib}^{\,b} = \vec{b}_g + (\mathcal{I} + M_g)\vec{\omega}_{ib}^{\,b} + G_g \vec{f}_{ib}^{\,b} + \vec{w}_g \tag{17}$$

Pos, Vel, Force and Angular Rate Errors

- Position error

$$\delta \vec{r}_{\beta b}^{\,\gamma} = \vec{r}_{\beta b}^{\,\gamma} - \hat{\vec{r}}_{\beta b}^{\,\gamma} \tag{18}$$

- Velocity error

$$\delta \vec{v}_{\beta b}^{\,\gamma} = \vec{v}_{\beta b}^{\,\gamma} - \hat{\vec{v}}_{\beta b}^{\,\gamma} \tag{19}$$

- Specific force errors

$$\delta \vec{f}_{ib}^{\,b} = \vec{f}_{ib}^{\,b} - \hat{\vec{f}}_{ib}^{\,b} \tag{20}$$

$$\Delta_e \vec{f}_{ib}^{\,b} = \Delta \vec{f}_{ib}^{\,b} - \Delta \hat{\vec{f}}_{ib}^{\,b} = -\delta \vec{f}_{ib}^{\,b} \tag{21}$$

- Angular rate errors

$$\delta \vec{\omega}_{ib}^{\,b} = \vec{\omega}_{ib}^{\,b} - \hat{\vec{\omega}}_{ib}^{\,b} \tag{22}$$

$$\Delta_e \vec{\omega}_{ib}^{\,b} = \Delta \vec{\omega}_{ib}^{\,b} - \Delta \hat{\vec{\omega}}_{ib}^{\,b} = -\delta \vec{\omega}_{ib}^{\,b} \tag{23}$$

D   Attitude Error

Attitude Error Definition

Define

$$\delta C_b^\gamma = C_b^\gamma \hat{C}_\gamma^b = e^{[\delta \vec{\psi}_{\gamma b}^{\,\gamma} \times]} \approx \mathcal{I} + [\delta \vec{\psi}_{\gamma b}^{\,\gamma} \times] \tag{24}$$

This is the error in attitude resulting from errors in estimating the angular rates.

The attitude error is a multiplicative small angle transformation from the actual frame to the computed frame

$$\hat{C}_b^\gamma = (\mathcal{I} - [\delta\vec{\psi}\,^\gamma_{\gamma b}\times])C_b^\gamma \tag{25}$$

Similarly,

$$C_b^\gamma = (\mathcal{I} + [\delta\vec{\psi}\,^\gamma_{\gamma b}\times])\hat{C}_b^\gamma \tag{26}$$

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E   Estimates of Sensor Measurements

Similarly we can attempt to estimate the specific force and angular rate by applying correction based on our estimate of the error.

$$\hat{\vec{f}}^b_{ib} = \tilde{\vec{f}}^b_{ib} - \Delta\hat{\vec{f}}^b_{ib} \tag{27}$$

$$\hat{\vec{\omega}}^b_{ib} = \tilde{\vec{\omega}}^b_{ib} - \Delta\hat{\vec{\omega}}^b_{ib} \tag{28}$$

where $\hat{\vec{f}}^b_{ib}$ and $\hat{\vec{\omega}}^b_{ib}$ are the accelerometer and gyroscope estimated calibration values, respectively.

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