EE 341

From the Fourier Series to the Fourier Transform

If $x(t)$ is not periodic, but is zero for $t < -T/2$ and $t > T/2$, then define $\tilde{x}(t)$ as $x(t)$ made periodic with period T — i.e., $\tilde{x}(t)$ is just $x(t)$ repeated every T.

Because $\tilde{x}(t)$ is periodic with period T, then $\omega_o = 2\pi/T$, and the Fourier series repesentation of $\tilde{x}(t)$ is:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_o t}$$

where

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_o t} dt$$

Change the name of a variable: let $\Delta\omega = \omega_o$ Then $T = 2\pi/\Delta\omega$.

$$c_k = \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\Delta\omega t} dt$$

and

$$\begin{aligned}
\tilde{x}(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\Delta\omega t} \\
&= \sum_{k=-\infty}^{\infty} \left\{ \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\Delta\omega t} dt \right\} e^{jk\Delta\omega t} \\
&= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \left\{ \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\Delta\omega t} dt \right\} e^{jk\Delta\omega t}\Delta\omega
\end{aligned}$$

Let

$$X(k\Delta\omega) = \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\Delta\omega t} dt$$

Then

$$\tilde{x}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(k\Delta\omega) e^{jk\Delta\omega t}\Delta\omega$$

Now take the limit as T goes to ∞. Then $\Delta\omega \to 0$, $k\Delta\omega$ becomes ω, the sum becomes an integral over ω, $\tilde{x}(t)$ becomes $x(t)$, and $X(k\Delta\omega)$ becomes $X(\omega)$:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

This is the Fourier transform for the aperiodic signal $x(t)$.